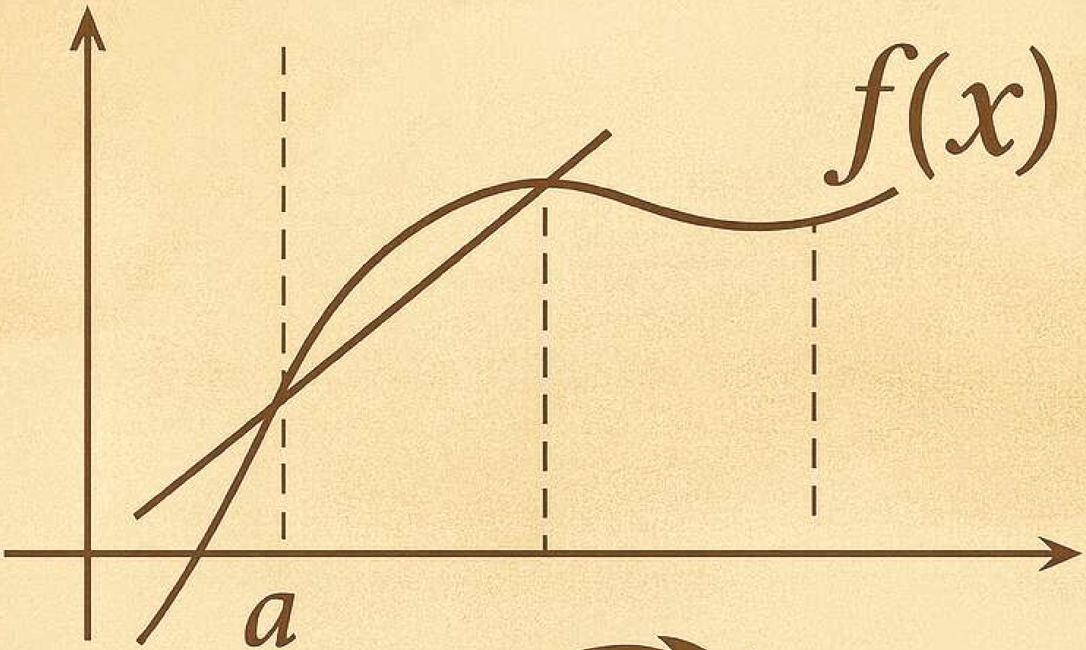
MATH106



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MATH106

1. Functions in Mathematics

Function Definition

A **function** is a rule that pairs every element in one set, called the **domain**, with exactly one element in another set, called the **range**. Formally, a function is a relation between two non-empty sets A and B, where each element in the domain A is assigned to exactly one element in the range B.

Function Notation

A function is typically denoted as:

$$f : A \rightarrow B$$

where f is the function, A is the domain, and B is the range. For any element $x \in A$, the function f assigns it to an element $f(x) \in B$.

For example, the function

$$f(x) = x^2 + 6$$

assigns every element x in the domain to $x^2 + 6$ in the range.

Function Notation with ":="

The symbol ":=" is sometimes used in function definitions, indicating that the function is being defined. For instance, if we write:

$$f(n) := 2n + 7$$

it means $f(n)$ is defined as $2n + 7$ for all n in the domain.

Domain, Range, and Co-domain

Domain

The **domain** of a function is the set of all possible input values. For example, in the function

$$f(x) = x^2 + 6$$

the domain could be the set of real numbers, integers, or another specified set.

Range

The **range** of a function is the set of all possible output values that the function can produce. In the case of

$$f(x) = x^2 + 6$$

the range depends on the domain. If the domain is the set of real numbers, the range would be all real numbers greater than or equal to 6.

Co-domain

The **co-domain** is the set in which the output values are considered to reside. The range is a subset of the co-domain.

Examples of Sets in Functions

- **Set of integers**:

$$Z = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

- **Set of positive integers**:

$$Z^+ = \{1, 2, 3, \ldots\}$$

- **Set of non-negative integers**:

$$Z_{\geq 0} = \{0, 1, 2, 3, \ldots\}$$

- **Set of rational numbers**:

$$Q$$

- **Set of real numbers**:

$$R$$

- **Intervals**:
  - Open interval:

$$(a, b) = x \in R \mid a < x < b$$

  - Half-open interval:

$$[a, b) = x \in R \mid a \leq x < b$$

  - Closed interval:

$$[a, b] = x \in R \mid a \leq x \leq b$$

Properties of Functions

1-to-1 Functions (Injective)

A function is called **1-to-1 (injective)** if every element in the domain maps to a unique element in the range. In other words, if $f(a) = f(b)$ then $a = b$.

Note: At most 1 x such that $f(x) = y$.

Onto Functions (Surjective)

A function is **onto (surjective)** if every element in the co-domain has a preimage in the domain. This means that for every $y \in B$, there exists an $x \in A$ such that $f(x) = y$.

Note: $range = codomain$

Into Functions

A function is **into** if not every element in the co-domain is mapped by the function. This means the range is a proper subset of the co-domain.

Note: $range \neq codomain$

Summary

- A function is a pairing between two sets, with each element in the domain paired with exactly one element in the range.
- Important concepts include domain, range, co-domain, and function properties like injective, surjective, and into.
- Notation is crucial in understanding and defining functions, with $f : A \to B$ as standard notation and ":=" used to define functions explicitly.

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2. Functions, Polynomials, and Trigonometric Functions

1-to-1 and Into Functions

1-to-1 (Injective) Function

A function f is called 1-to-1 or injective if different inputs produce different outputs, i.e., if $f(x_1) = f(x_2)$, then $x_1 = x_2$. In other words, each element of the domain maps to a unique element in the codomain.

Into Function (Surjective)

A function f is called an into or surjective function if every element in the codomain (output set) has at least one preimage in the domain. This means that for every y in the codomain, there exists an x in the domain such that $f(x) = y$.

Both 1-to-1 and Onto (Bijective) Function

A function that is both injective and surjective is called a bijective function. This type of function has an inverse because every element in the codomain is matched uniquely with an element in the domain.

Composite Functions

A composite function is created when one function is applied to the result of another function. If f and g are two functions, then the composite function $f(g(x))$ means that the function g is applied first and then the function f is applied to the result of $g(x)$.

Notation

The composite of f and g is written as:

$$(f \circ g)(x) = f(g(x))$$

This reads as "f composed with g of x."

Inverse Functions

An inverse function reverses the operation of a function. If $f(x)$ is a function, its inverse $f^{-1}(x)$ satisfies the following condition:

$$f(f^{-1}(x)) = f^{-1}(f(x)) = x$$

This means that applying a function followed by its inverse (or vice versa) will return the original input.

Example

If $f(x) = 2x + 3$, then the inverse function $f^{-1}(x)$ can be found by solving for x in terms of y:

$$y = 2x + 3 \implies x = \frac{y-3}{2}$$

Thus, $f^{-1}(x) = \frac{x-3}{2}$.

Polynomials

A polynomial $p(x)$ is a function of the form:

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

Where $a_n, a_{n-1}, \ldots, a_0$ are constants, and n is the degree of the polynomial.

Key Terms:

- **Constant Coefficient**: The term a_0 is the constant term in the polynomial.
- **Leading Coefficient**: The coefficient a_n of the highest degree term is called the leading coefficient.
- **Degree of a Polynomial (deg(p))**: The degree of the polynomial is the highest exponent of x that appears in the polynomial.

Roots of a Polynomial

A root (or zero) of a polynomial is a solution to the equation $p(x) = 0$ If r is a root, then:

$$p(r) = 0$$

Discriminant and Roots of Quadratic Polynomials

For a quadratic polynomial $ax^2 + bx + c = 0$, the discriminant Δ is given by:

$$\Delta = b^2 - 4ac$$

- If $\Delta > 0$, the quadratic has two distinct real roots.
- If $\Delta = 0$, the quadratic has one real root (a repeated root).
- If $\Delta < 0$, the quadratic has two complex roots.

Rational Functions

A rational function is the ratio of two polynomials:

$$r(x) = \frac{p(x)}{q(x)}$$

Where $p(x)$ and $q(x)$ are polynomials, and $q(x) \neq 0$.

Domain and Range of Rational Functions

- **Domain**: The domain of a rational function excludes the values of x that make the denominator $q(x) = 0$.
- **Range**: The set of possible output values of the function.

Degree of a Rational Function

The degree of a rational function $r(x) = \frac{p(x)}{q(x)}$ is:

$$\deg(r(x)) = \deg(p) - \deg(q)$$

This degree can be negative if the degree of the denominator $q(x)$ is larger than the degree of the numerator $p(x)$.

Note: Rational functions can have negative degrees, unlike polynomials.

Trigonometric Functions

Trigonometric functions can be defined using the unit circle, which is a circle centered at the origin with a radius of 1 in the coordinate plane.

Unit Circle Definition

For an angle θ measured from the positive x-axis (in radians), the trigonometric functions can be understood as the coordinates of a point on the unit circle.

- **Sine**: $\sin(\theta)$ represents the y-coordinate of the point on the unit circle at angle θ.
- **Cosine**: $\cos(\theta)$ represents the x-coordinate of the point on the unit circle at angle θ.

Unit Circle

The unit circle is a circle with a radius of 1 centered at the origin. The coordinates of a point on the unit circle are $(\cos(\theta), \sin(\theta))$, where θ is the angle from the positive x-axis.

Other Trigonometric Functions

- **Tangent**: $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$
- **Cotangent**: $\cot(\theta) = \frac{1}{\tan(\theta)}$

- **Secant**: $\sec(\theta) = \frac{1}{\cos(\theta)}$
- **Cosecant**: $\csc(\theta) = \frac{1}{\sin(\theta)}$

Domains and Ranges of Trigonometric Functions

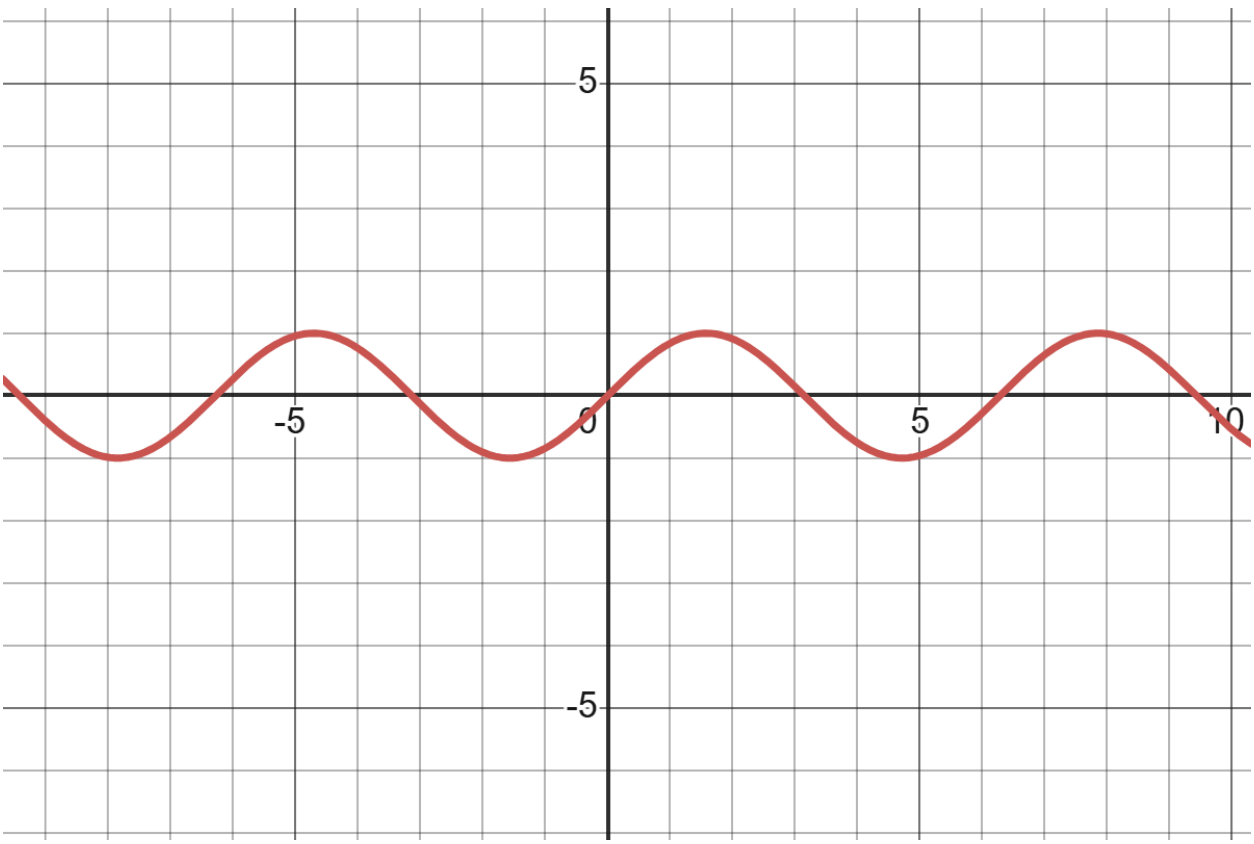
1. Sine Function $\sin(x)$

- **Domain**: The sine function is defined for all real numbers. Therefore, the domain of $\sin(x)$ is:

$$(-\infty, \infty)$$

- **Range**: The sine function oscillates between -1 and 1. Thus, the range of $\sin(x)$ is:

$$[-1, 1]$$



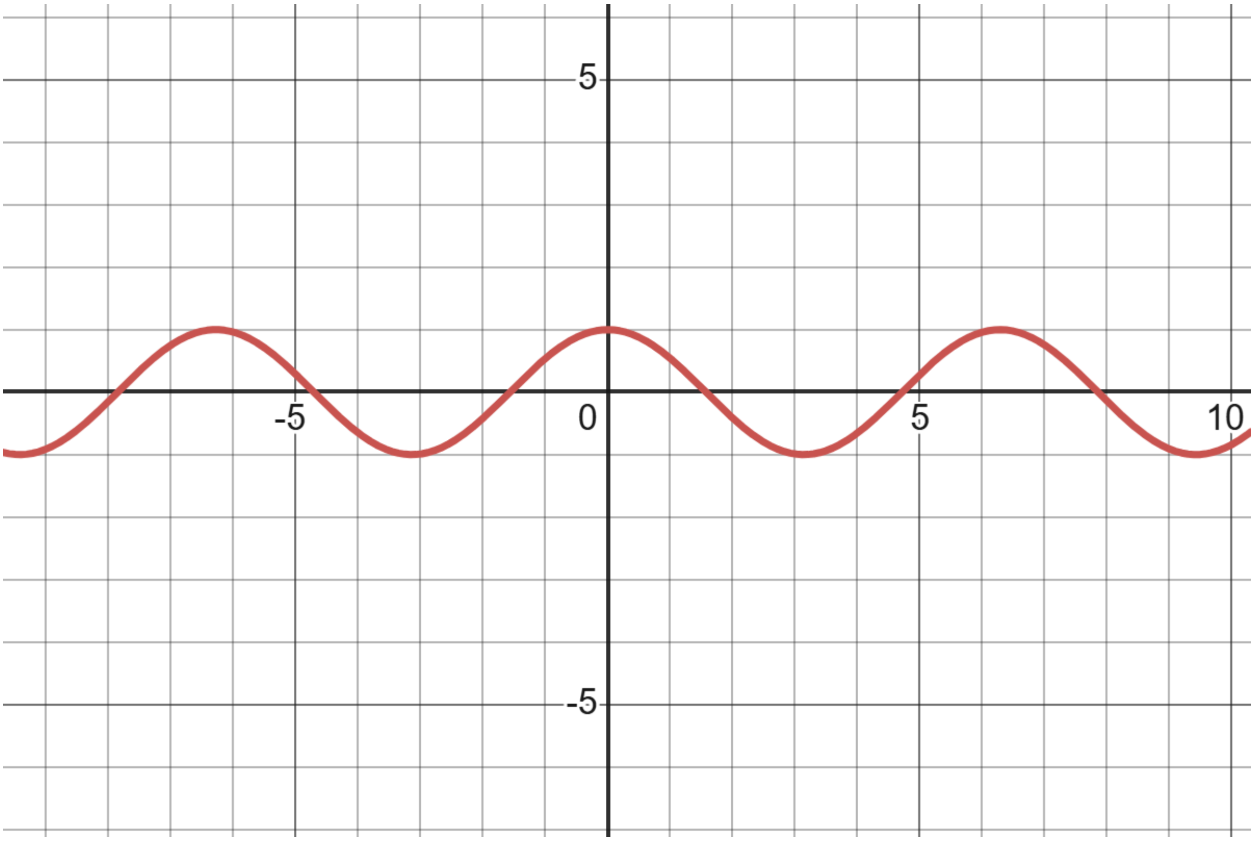
2. Cosine Function $\cos(x)$

- **Domain**: The cosine function is also defined for all real numbers. Therefore, the domain of $\cos(x)$ is:

$$(-\infty, \infty)$$

- **Range**: The cosine function oscillates between -1 and 1. Thus, the range of $\cos(x)$ is:

$$[-1, 1]$$



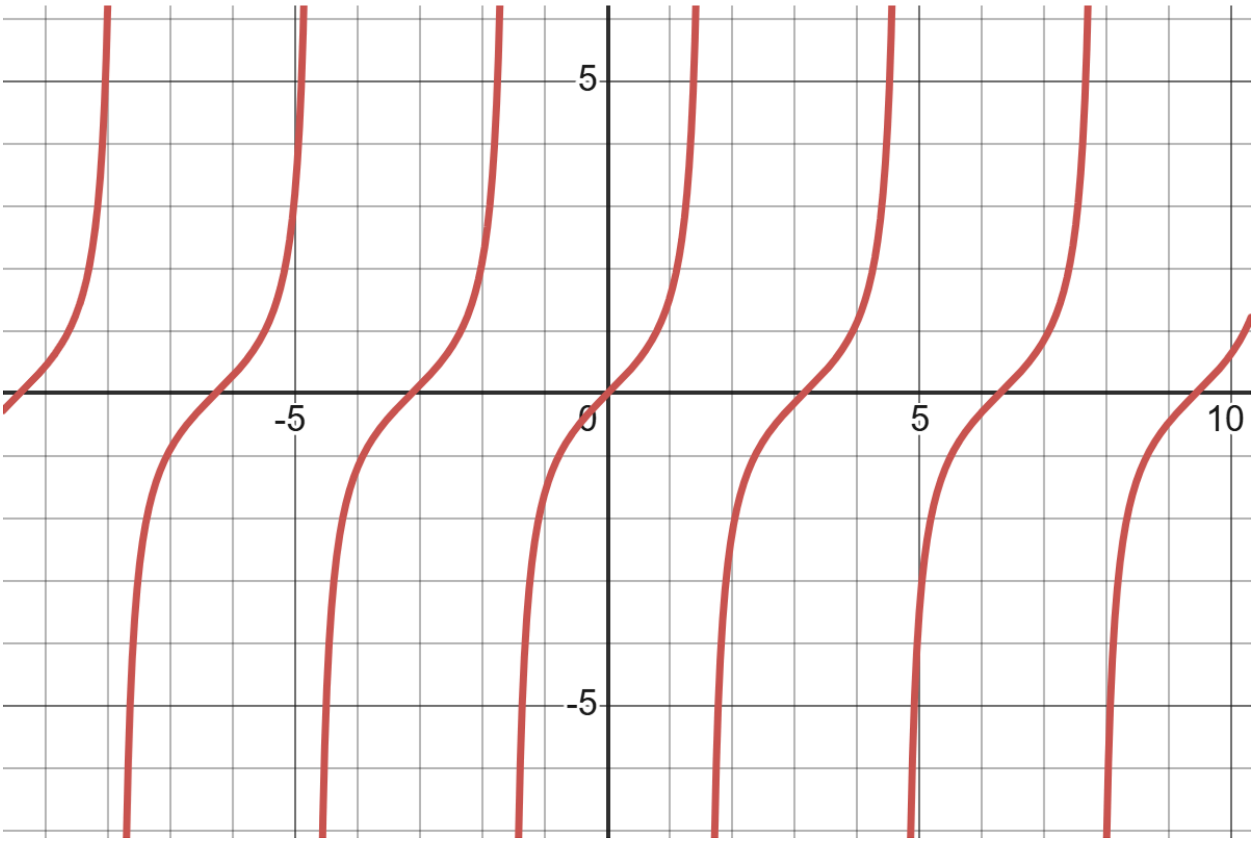
3. **Tangent Function** $\tan(x)$

- **Domain**: The tangent function is undefined at odd multiples of $\frac{\pi}{2}$, because the cosine function in the denominator equals 0 at these points. Therefore, the domain of $\tan(x)$ is:

$$(-\infty, \infty) \quad \text{except} \quad x \neq \frac{(2n+1)\pi}{2}, n \in \mathbb{Z}$$

- **Range**: The tangent function has a range of all real numbers, as it increases without bound:

$$(-\infty, \infty)$$



4. Cotangent Function $\cot(x)$

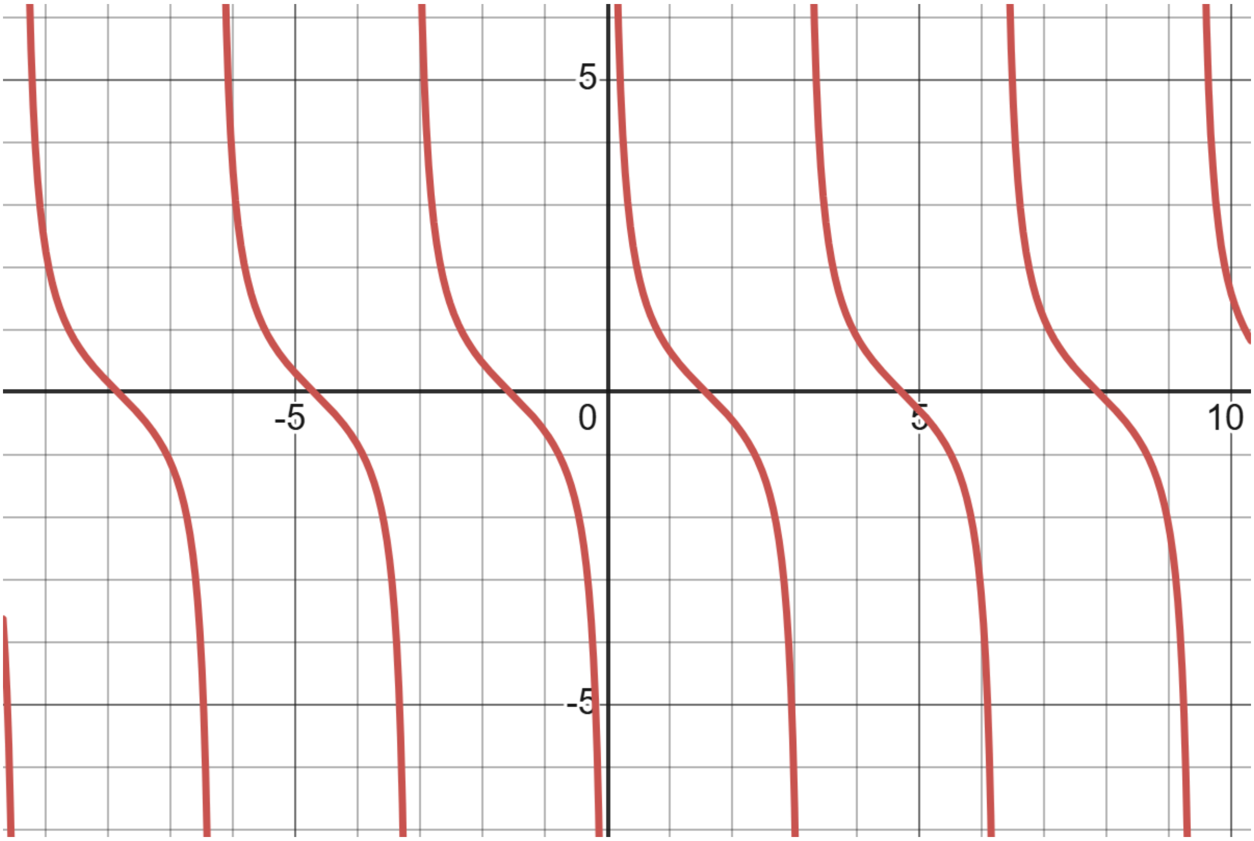
- **Domain**: The cotangent function is undefined where $\sin(x) = 0$, at integer multiples of π. Therefore, the domain of $\cot(x)$ is:

$$(-\infty, \infty) \quad \text{except} \quad x \neq n\pi, \, n \in \mathbb{Z}$$

- **Range**: The range of $\cot(x)$ is:

$$(-\infty, \infty)$$

  because it behaves similarly to the tangent function, with vertical asymptotes at $x = n\pi$.



5. **Secant Function** $\sec(x)$

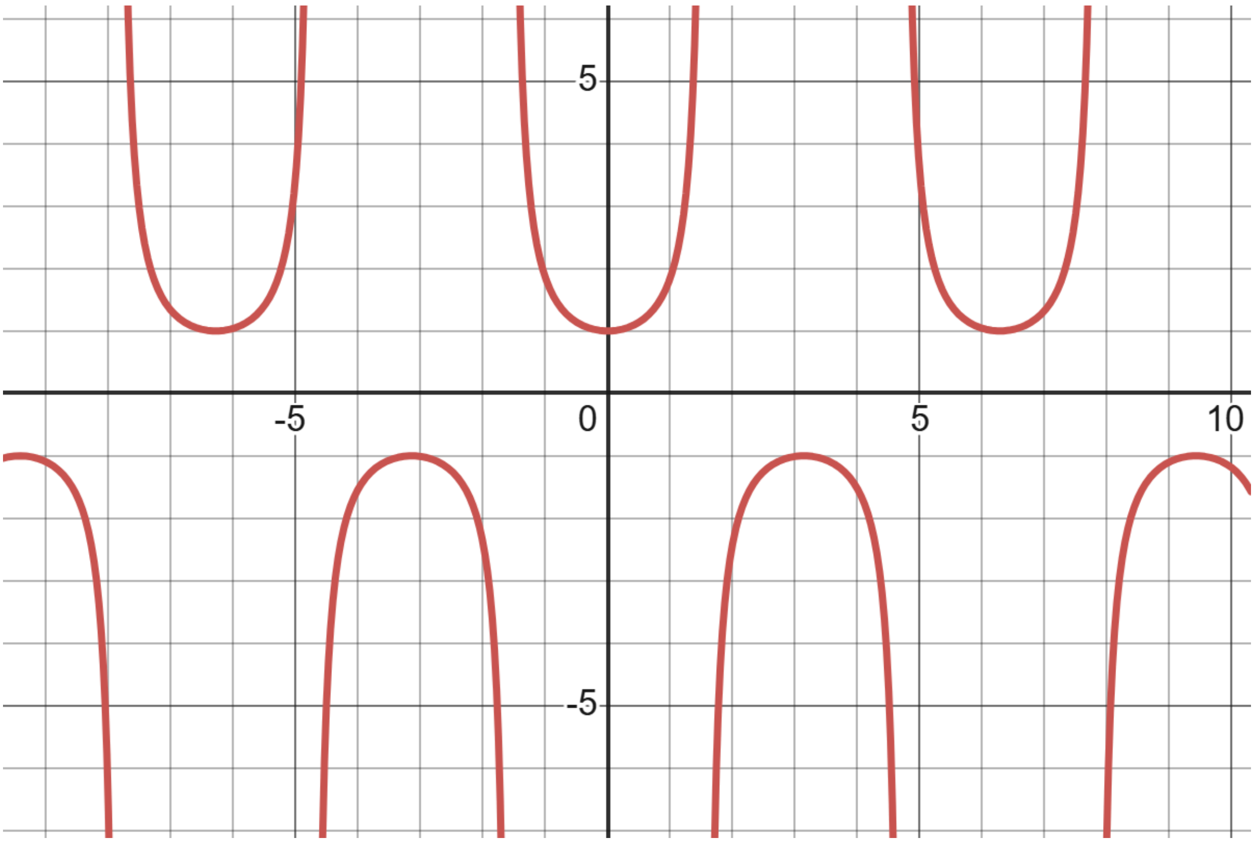
- **Domain**: The secant function is undefined where $\cos(x) = 0$, which occurs at odd multiples of $\frac{\pi}{2}$. Therefore, the domain of $\sec(x)$ is:

$$(-\infty, \infty) \quad \text{except} \quad x \neq \frac{(2n+1)\pi}{2}, \, n \in \mathbb{Z}$$

- **Range**: The range of $\sec(x)$ is:

$$(-\infty, -1] \cup [1, \infty)$$

because $\sec(x) = \frac{1}{\cos(x)}$, and $\cos(x)$ is between -1 and 1, excluding 0.



6. Cosecant Function $\csc(x)$

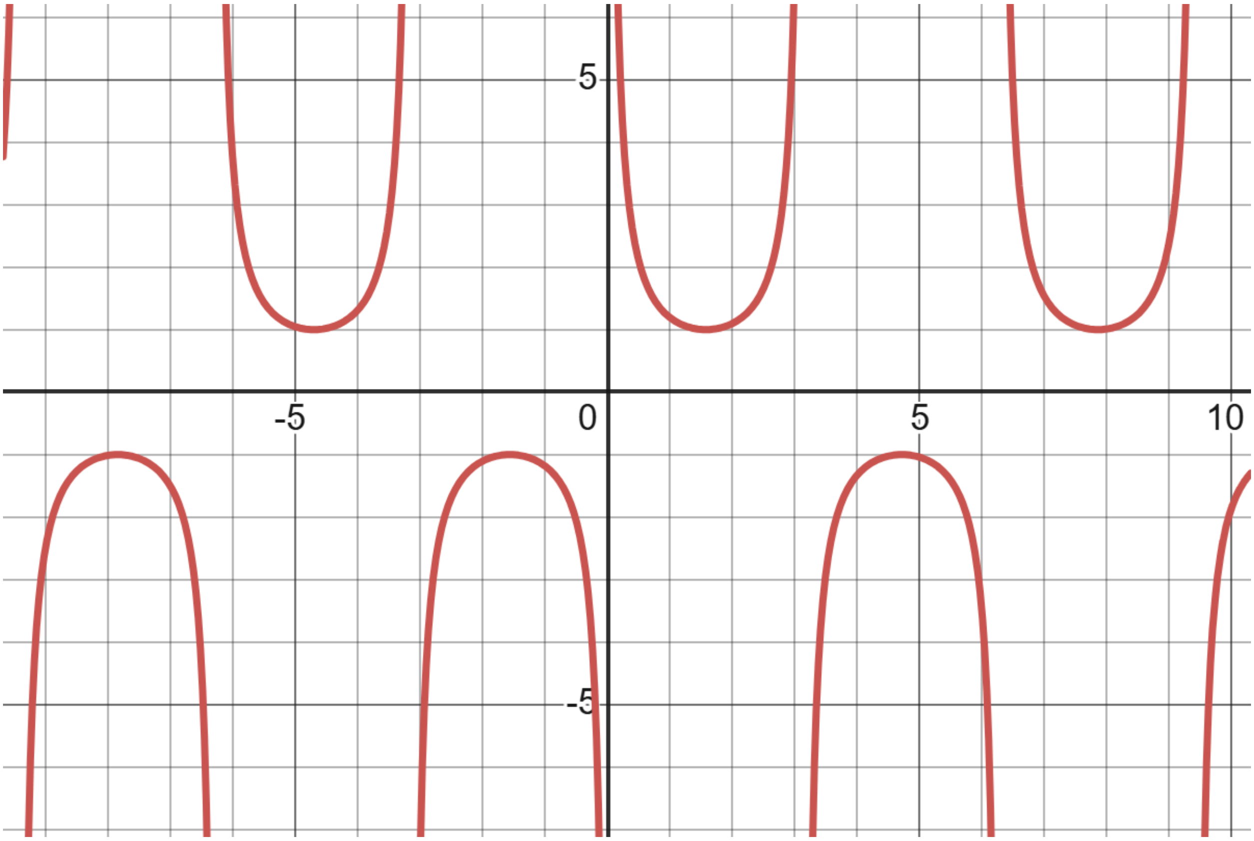
- **Domain**: The cosecant function is undefined where $\sin(x) = 0$. This happens at integer multiples of π. Therefore, the domain of $\csc(x)$ is:

$$(-\infty, \infty) \quad \text{except} \quad x \neq n\pi,\, n \in \mathbb{Z}$$

- **Range**: The range of $\csc(x)$ is:

$$(-\infty, -1] \cup [1, \infty)$$

since $\csc(x) = \frac{1}{\sin(x)}$ and $\sin(x)$ is between -1 and 1, excluding 0.



Inverse Trigonometric Functions

The inverse trigonometric functions return the angle whose trigonometric ratio is the given value. For example:

- $\arcsin(x)$ is the angle θ such that $\sin(\theta) = x$.
- $\arccos(x)$ is the angle θ such that $\cos(\theta) = x$.

1-to-1 and Onto Properties of Trigonometric Functions

- **Sine**: $\sin(\theta)$ is 1-to-1 and onto in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$, and its inverse $\arcsin(x)$ is defined in this range.
- **Cosine**: $\cos(\theta)$ is 1-to-1 and onto in the interval $[0, \pi]$, and its inverse $\arccos(x)$ is defined in this range.

3. Introduction to Limits and Continuity

Informal Definition of Limit

A **limit** describes the behavior of a function as the input x approaches a particular value. We are interested in what happens to $f(x)$ as x gets close to a certain value, rather than the exact value at that point.

Definition:

Let $f(x)$ be a function. The limit of $f(x)$ as x approaches a is denoted by $\lim_{x \to a} f(x) = L$, which means:

- $f(x)$ approaches the value L as x gets arbitrarily close to a from both sides (left and right), without necessarily evaluating $f(a)$.

Left and Right Limits

For a given real number a, the limit can approach from two directions:

- **Left-hand limit**: If $f(x)$ approaches L as x approaches a from the left (i.e., $x < a$), we write:

$$\lim_{x \to a^-} f(x) = L$$

- **Right-hand limit**: If $f(x)$ approaches L as x approaches a from the right (i.e., $x > a$), we write:

$$\lim_{x \to a^+} f(x) = L$$

If the left-hand and right-hand limits are equal, the overall limit exists:

$$\lim_{x \to a} f(x) = L$$

If the two are not equal, the limit does not exist.

Terminology

Neighborhood and Deleted Neighborhood

- **Neighborhood of a**: This refers to an open interval containing a. For example, $(a - \delta, a + \delta)$ is a neighborhood of a, where δ is a small positive number.
- **Deleted Neighborhood of a**: This is an open interval around a, excluding a itself. It can be written as $(b, c) \setminus \{a\}$ or $(b, a) \cup (a, c)$ meaning the interval does not include the point a, though it approaches it from both sides.

Formal Definition of Limit

For a function $f(x)$, the limit of $f(x)$ as x approaches a is L, written $\lim_{x \to a} f(x) = L$, if:

- For every $\epsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$.

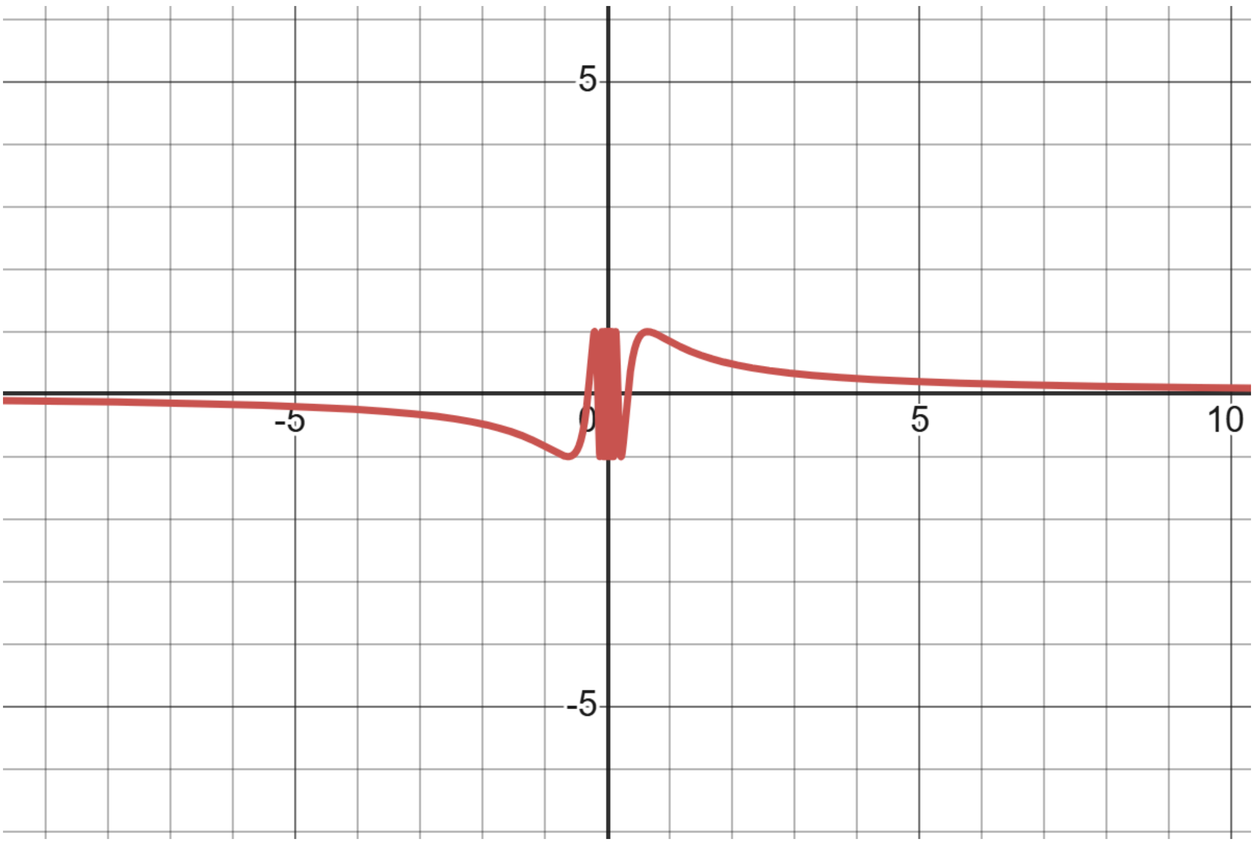
This formal definition captures the idea that $f(x)$ gets arbitrarily close to L as x approaches a, regardless of how close x is to a.

Examples of Limits

Example 1: $f(x) = \sin\left(\frac{1}{x}\right)$

As x approaches 0, $f(x) = \sin\left(\frac{1}{x}\right)$ oscillates infinitely between -1 and 1. Since $f(x)$ does not settle towards any particular value as $x \to 0$, the limit does not exist.

$$\lim_{x \to 0} f(x) = d.n.e$$



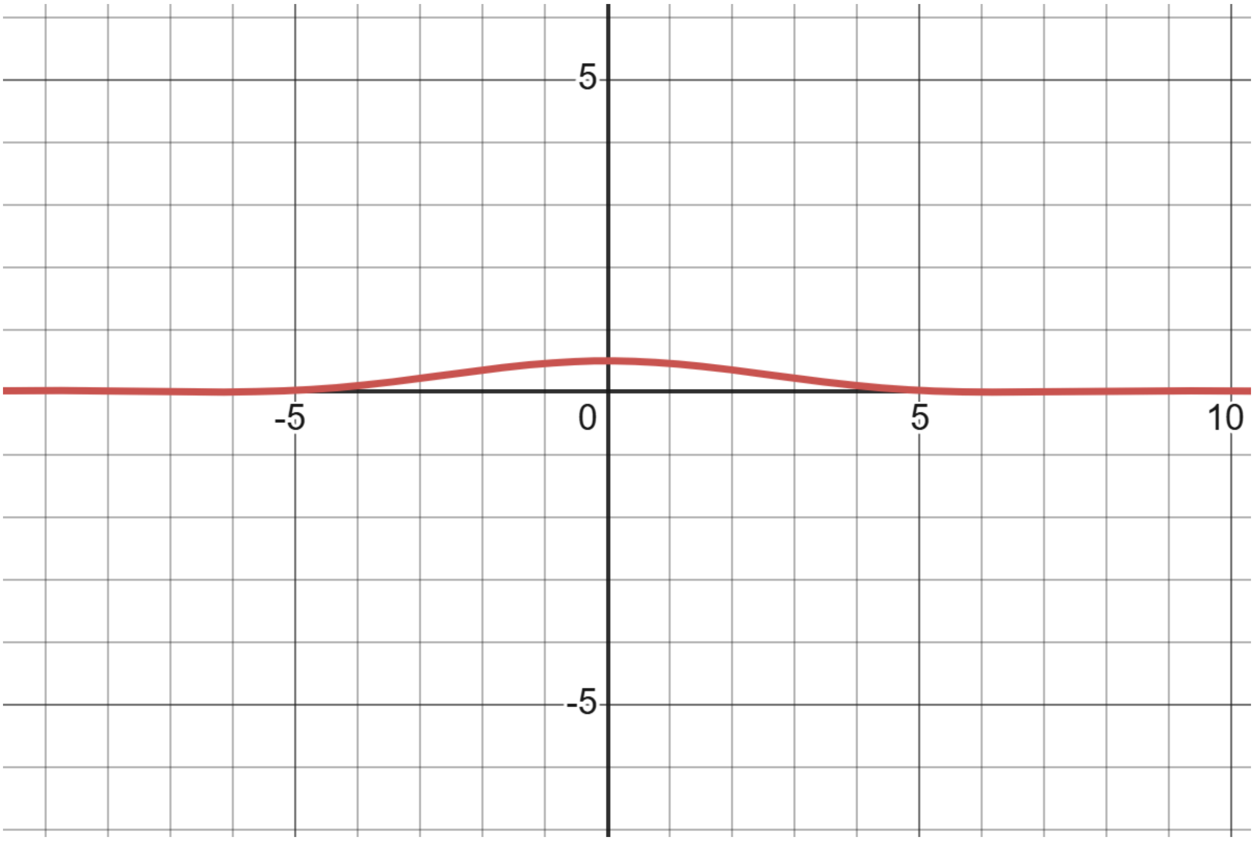
Example 2: $g(x) = \frac{1-\cos(x)}{x^2}$

As $x \to 0$, the function $g(x) = \frac{1-\cos(x)}{x^2}$ is continuous for all $x \neq 0$. Evaluating the limit by substituting small values of x, we observe that the limit approaches a finite value.

Thus, the limit as $x \to 0$ exists:

$$\lim_{x \to 0} g(x) = \frac{1}{2}.$$

The function is continuous everywhere except a $x = 0$.

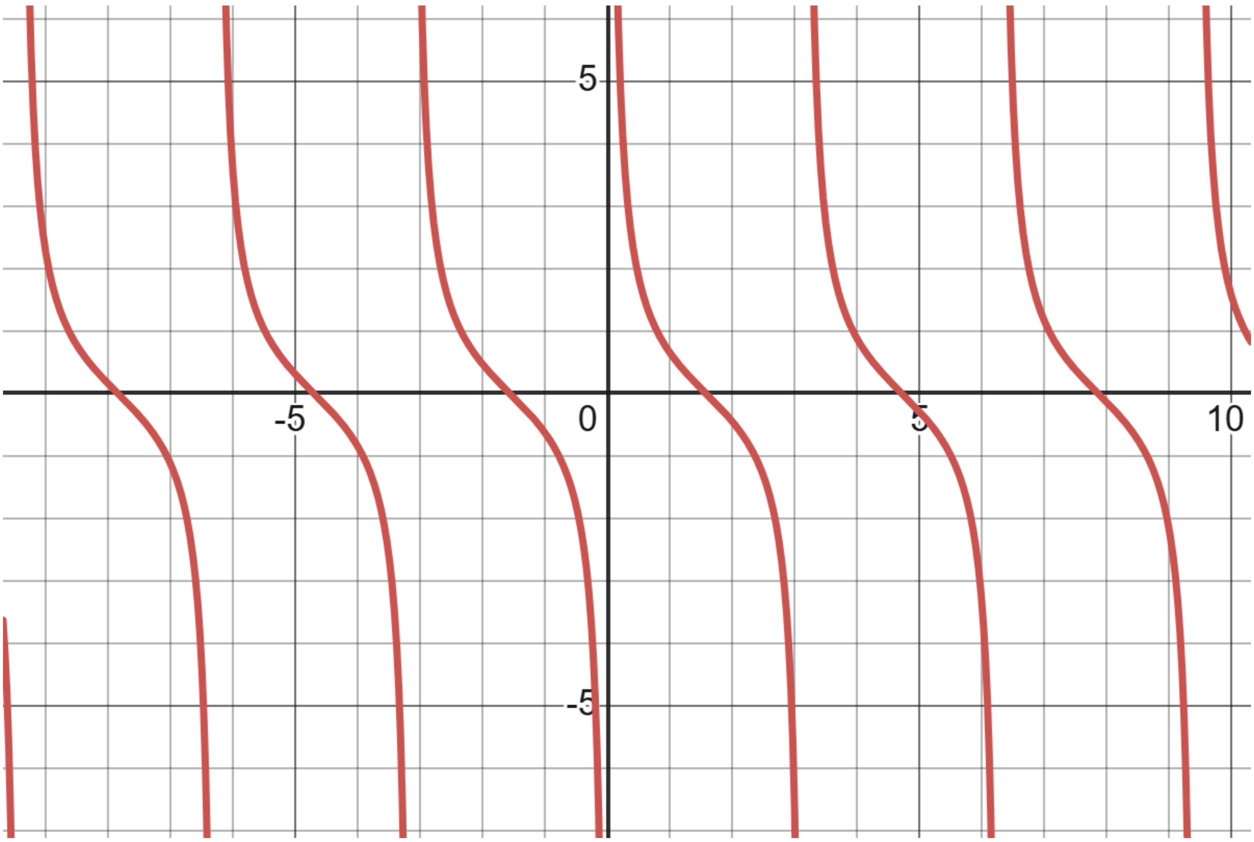


Example 3: $h(x) = \cot(x)$

The function $h(x) = \cot(x) = \frac{\cos(x)}{\sin(x)}$ is undefined at $x = n\pi$ where $\sin(x) = 0$. The limit does not exist at these points due to vertical asymptotes. For example, at $x = \pi$, the limit as x approaches from the left or right is infinite.

$$\lim_{x \to \pi^-} h(x) = -\infty, \lim_{x \to \pi^+} h(x) = +\infty$$

Thus, $h(x)$ is continuous everywhere except at $x = n\pi$.



Summary of Key Concepts

- **Limit**: The value that a function approaches as the input approaches a particular point.
- **Left-hand limit**: The limit as x approaches from the left.
- **Right-hand limit**: The limit as x approaches from the right.
- **Neighborhood**: An open interval around a point.
- **Deleted Neighborhood**: An open interval around a point, excluding the point itself.
- **Oscillation**: A situation where a function does not approach a single value as x approaches a point (e.g., $\sin(1/x)$).

4. Limits and Their Properties

Example: Formal Definition of Limit

Let $f(x)$ be a function. Consider the formal definition of a limit.

$\lim_{x \to a} f(x) = L$ means that as x approaches a, the values of $f(x)$ get arbitrarily close to L.

Example

Let $f(x) = 2x + 1$, and we are interested in finding $\lim_{x \to 1} f(x)$.

Applying the definition of the limit:

$$\lim_{x \to 1}(2x + 1) = 2(1) + 1 = 3$$

Thus, the limit exists and equals 3.

Properties of Limits

Given two functions $f(x)$ and $g(x)$, and let $a \in \mathbb{R}$, such that:

- $\lim_{x \to a} f(x) = L$
- $\lim_{x \to a} g(x) = M$

Then the following properties hold:

1. **Addition/Subtraction**

$$\lim_{x \to a}[f(x) \pm g(x)] = L \pm M$$

2. **Multiplication**

$$\lim_{x \to a}[f(x) \cdot g(x)] = L \cdot M$$

3. **Division** (provided $M \neq 0$)

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{M}$$

Example: Limit of a Polynomial

Consider $f(x) = x^2 - 2x + 3$ and find $\lim_{x \to a} f(x)$.

Using the properties of limits, compute the limit of each term:

$$\lim_{x \to a}(x^2 - 2x + 3) = \lim_{x \to a} x^2 - \lim_{x \to a} 2x + \lim_{x \to a} 3$$

Substituting the limit values:

$$= a^2 - 2a + 3$$

Thus, $\lim_{x \to a}(x^2 - 2x + 3) = a^2 - 2a + 3$.

Remark: Limits of Polynomials

If $p(x)$ is a polynomial, then for any $a \in \mathbb{R}$,

$$\lim_{x \to a} p(x) = p(a)$$

This means that you can find the limit of a polynomial at any point by simply substituting a into the polynomial.

Formal Definition of Limit Using Left and Right Limits

Let $f(x)$ be a function and $a \in \mathbb{R}$. The limit of $f(x)$ as $x \to a$ is L if and only if:

- The left-hand limit:

$$\lim_{x \to a^-} f(x) = L$$

- The right-hand limit:

$$\lim_{x \to a^+} f(x) = L$$

If the left and right limits exist and are equal, then:

$$\lim_{x \to a} f(x) = L$$

If the left and right limits do not exist or are not equal, the overall limit does not exist (d.n.e.).

Piecewise Defined Functions and Limits

For piecewise-defined functions, you evaluate the left and right limits at the point where the function changes definition.

Example

Let $f(x) = \begin{cases} x+1 & \text{if } x < 1 \\ 2x-1 & \text{if } x \geq 1 \end{cases}$

To find $\lim_{x \to 1} f(x)$:

- Left limit:

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x+1) = 1 + 1 = 2$$

- Right limit:

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (2x-1) = 2(1) - 1 = 1$$

Since the left and right limits are not equal, $\lim_{x \to 1} f(x)$ does not exist.

Fundamental Fact

The limit $\lim_{x \to a} f(x) = L$ if and only if the left-hand and right-hand limits exist and are equal to each other. If they are unequal, the limit does not exist.

Homework

Prove: If $f(x) = c$ (a constant function), then:

$\lim_{x \to a} f(x) = c$ for any $a \in \mathbb{R}$.

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5. Continuity and Limits of Functions

Continuity: Definition

Let f be a function defined in a neighborhood of a in real numbers (meaning a is within the domain of f). We say that f is **continuous** at $x = a$ if:

$$\lim_{x \to a} f(x) = f(a)$$

This implies that the limit of $f(x)$ as x approaches a equals the value of f at $x = a$.

Examples of Continuity

1. **Constant Function**

   Let $f(x) = \lambda x$. We have:

   $$\lim_{x \to a} \lambda x = \lambda a = f(a)$$

   So, f is continuous at $x = a$ for any real a.

2. **Polynomials**

   For any polynomial $p(x)$, the limit as x approaches any real a is:

   $$\lim_{x \to a} p(x) = p(a)$$

   Therefore, all **polynomials** are continuous at any point a in the real numbers.

Remarks on Continuity of Special Functions

- **Rational Functions**: All rational functions are continuous on their domains. For example:

$$f(x) = \frac{x - 1}{x^2 - 4}$$

Here, the domain of f is $\mathbb{R} \setminus \{2, -2\}$, and f is continuous at every point within its domain.

Note: For points outside of the domain (e.g., $x = 2$ or $x = -2$), continuity does not apply.

- **Trigonometric Functions**: All trigonometric functions are continuous within their domains.
- **Inverse Trigonometric Functions**: All inverse trigonometric functions are also continuous on their domains.

Properties of Continuous Functions

Let f and g be two continuous functions and let $a \in \mathbb{R}$. Then:

1. **Addition**: $\lim_{x\to a}(f(x) + g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
2. **Subtraction**: $\lim_{x\to a}(f(x) - g(x)) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
3. **Multiplication**: $\lim_{x\to a}(f(x) \cdot g(x)) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$
4. **Division**: $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ provided $\lim_{x\to a} g(x) \neq 0$.

Example of Continuity

$$\lim_{x\to 0} \frac{\sin^2(x) + \sin(x) + 1}{\cos(x)} = \frac{\sin^2(0) + \sin(0) + 1}{\cos(0)} = \frac{0 + 0 + 1}{1} = \frac{1}{1} = 1.$$

Thus,

$$\lim_{x\to 0} \frac{\sin^2(x) + \sin(x) + 1}{\cos(x)} = 1.$$

Floor and Ceiling Functions

1. **Floor Function** $\lfloor x \rfloor$: Maps x to the greatest integer less than or equal to x.
2. **Ceiling Function** $\lceil x \rceil$: Maps x to the smallest integer greater than or equal to x.

Examples:

- $\lfloor 2.3 \rfloor = 2$
- $\lceil 2.3 \rceil = 3$

Infinite Limits: Definition

Let f be a function defined in a deleted neighborhood of a in the real numbers. We say that the **limit of f as x approaches a is infinity**, and we write:

$$\lim_{x\to a} f(x) = +\infty$$

if for any $M > 0$ there exists some $\delta > 0$ such that $0 < |x - a| < \delta \Rightarrow f(x) > M$.

Similarly, we write:

$$\lim_{x\to a} f(x) = -\infty$$

if for any $m < 0$ there exists some $\delta > 0$ such that $0 < |x - a| < \delta \Rightarrow f(x) < m$.

Homework

Problem: Define $\lim_{x \to a^-} f(x) = \pm\infty$.

Solution:

1. **For** $+\infty$:

$$\lim_{x \to a^-} f(x) = +\infty \quad \text{if for every } M > 0, \text{ there exists a } \delta > 0 \text{ such that if } 0 < a - x < \delta, \text{ then } f(x) > M.$$

1. **For** $-\infty$:

$$\lim_{x \to a^-} f(x) = -\infty \quad \text{if for every } N < 0, \text{ there exists a } \delta > 0 \text{ such that if } 0 < a - x < \delta, \text{ then } f(x) < N.$$

Interpretation:

- $+\infty$: As x approaches a from the left, $f(x)$ exceeds any positive number M.
- $-\infty$: As x approaches a from the left, $f(x)$ falls below any negative number N.

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6. Limits at Infinity, Infinite Limits, and Squeeze Theorem

Limits at Infinity

Definition

The limit at infinity describes the behavior of a function $f(x)$ as x approaches $+\infty$ or $-\infty$. When we write:

$$\lim_{x \to \infty} f(x) = L \quad \text{or} \quad \lim_{x \to -\infty} f(x) = L$$

it means that as x becomes very large in the positive or negative direction, $f(x)$ approaches a particular finite value L .

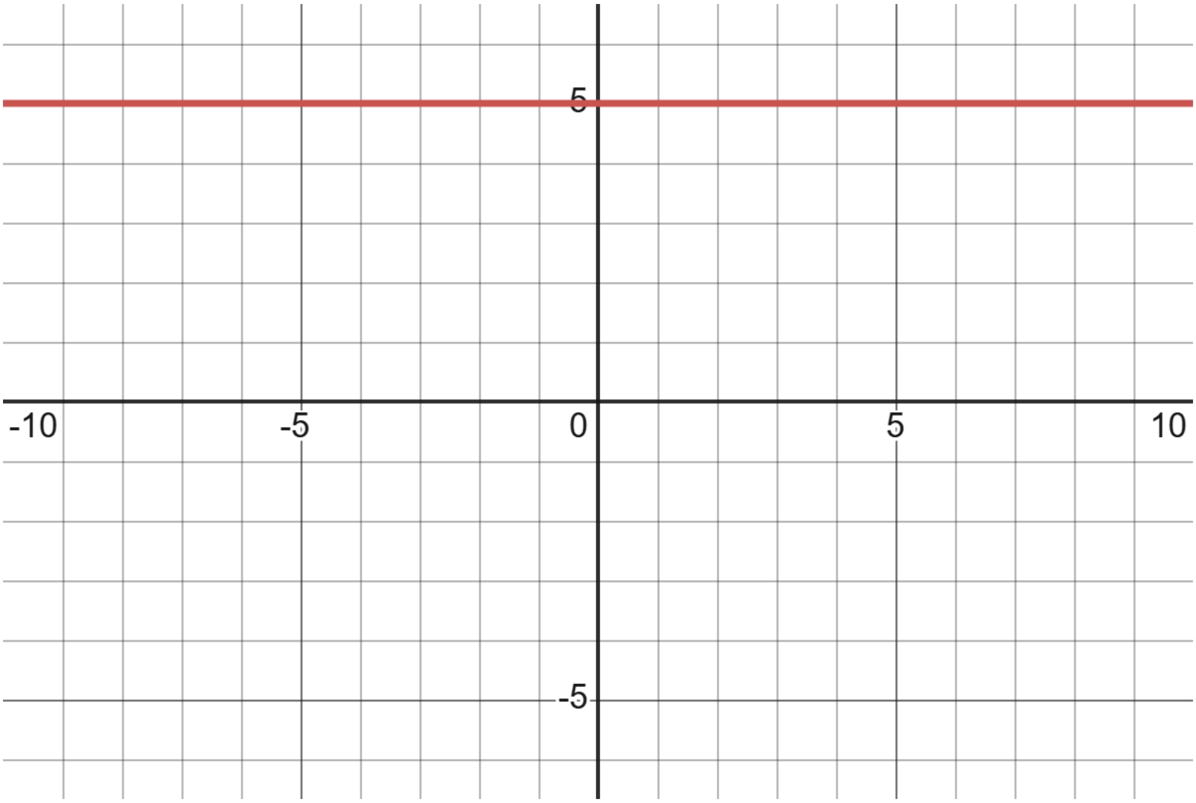
Examples

1. **Constant function**:

   For a constant function like $f(x) = 5$:

$$\lim_{x \to \infty} 5 = 5$$

Here, $f(x)$ stays constant at 5 as $x \to \infty$.

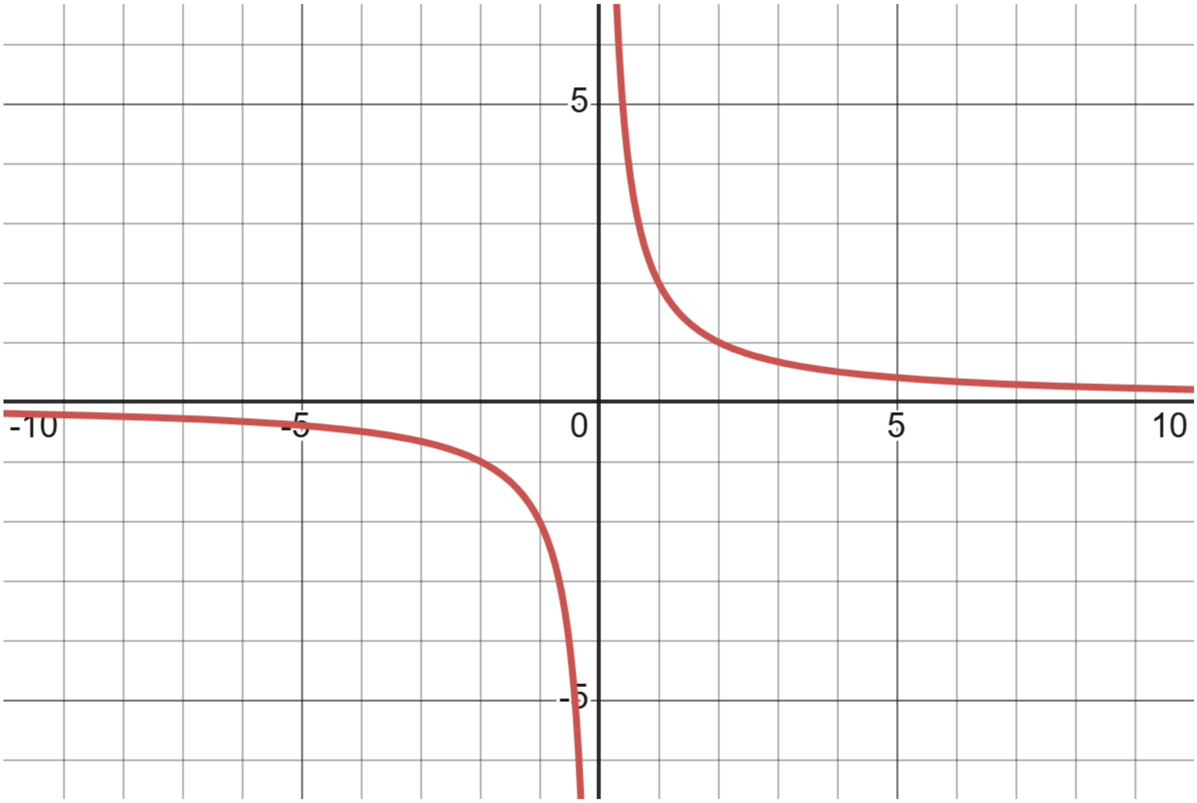


2. **Rational function with a degree comparison**:

   For $f(x) = \frac{2}{x}$:

$$\lim_{x \to \infty} \frac{2}{x} = 0$$

   As $x \to \infty$, $\frac{2}{x}$ gets smaller and approaches 0.

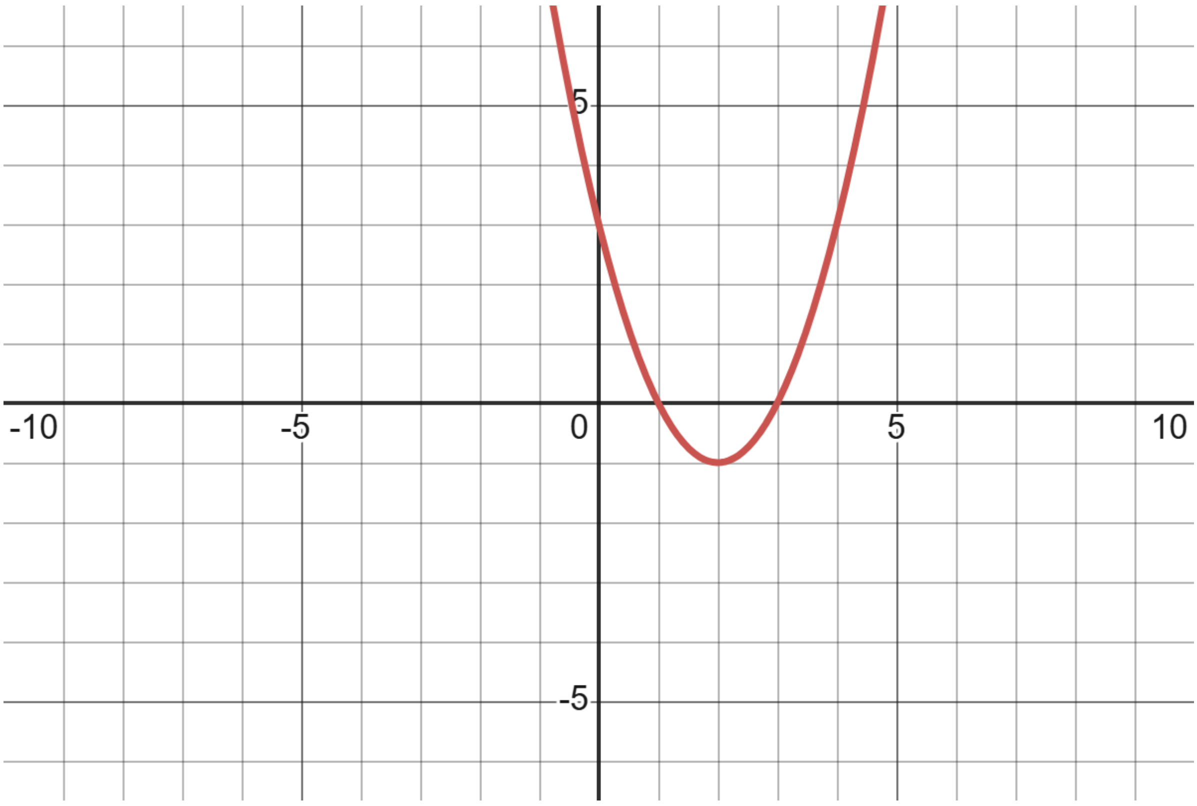


3. **Polynomial function**:

   For $f(x) = x^2 - 4x + 3$:

$$\lim_{x \to \infty} (x^2 - 4x + 3) = \infty$$

   Since x^2 dominates as x becomes very large, $f(x)$ approaches $+\infty$.

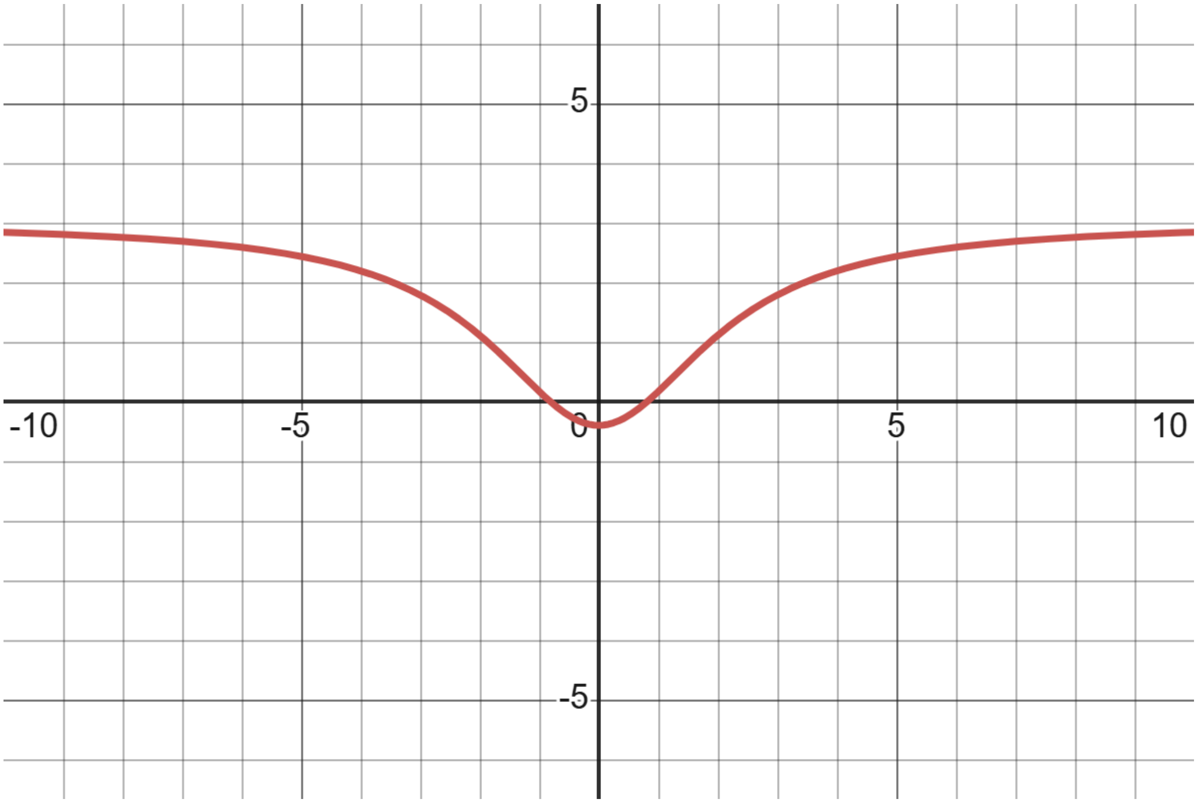


4. **Rational function example with degrees equal**:

   For $f(x) = \frac{3x^2-2}{x^2+5}$, where the degrees of the numerator and denominator are equal:

   $$\lim_{x \to \infty} \frac{3x^2 - 2}{x^2 + 5} = \frac{3}{1} = 3$$

   This is because we focus on the leading coefficients when the degrees are equal.

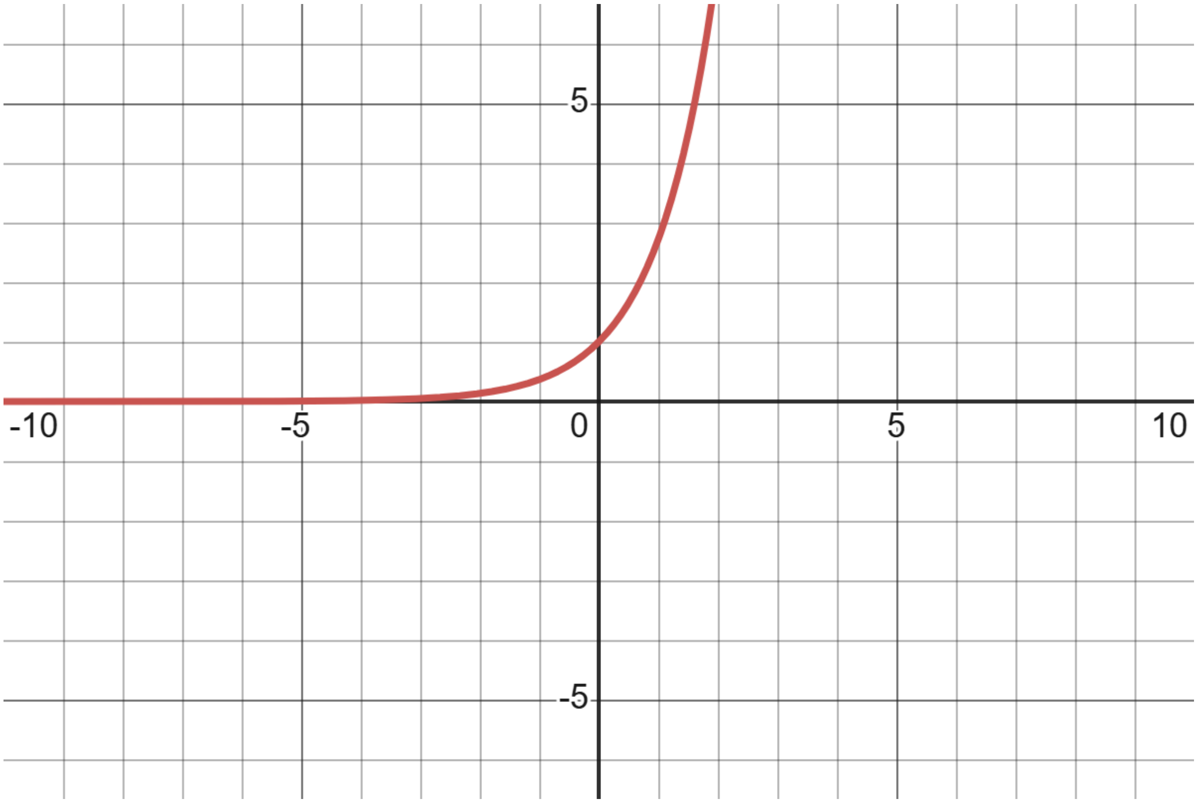


5. **Exponential function**:

   For $f(x) = e^x$:

$$\lim_{x \to \infty} e^x = \infty$$

   Here, e^x grows rapidly to infinity as $x \to \infty$.

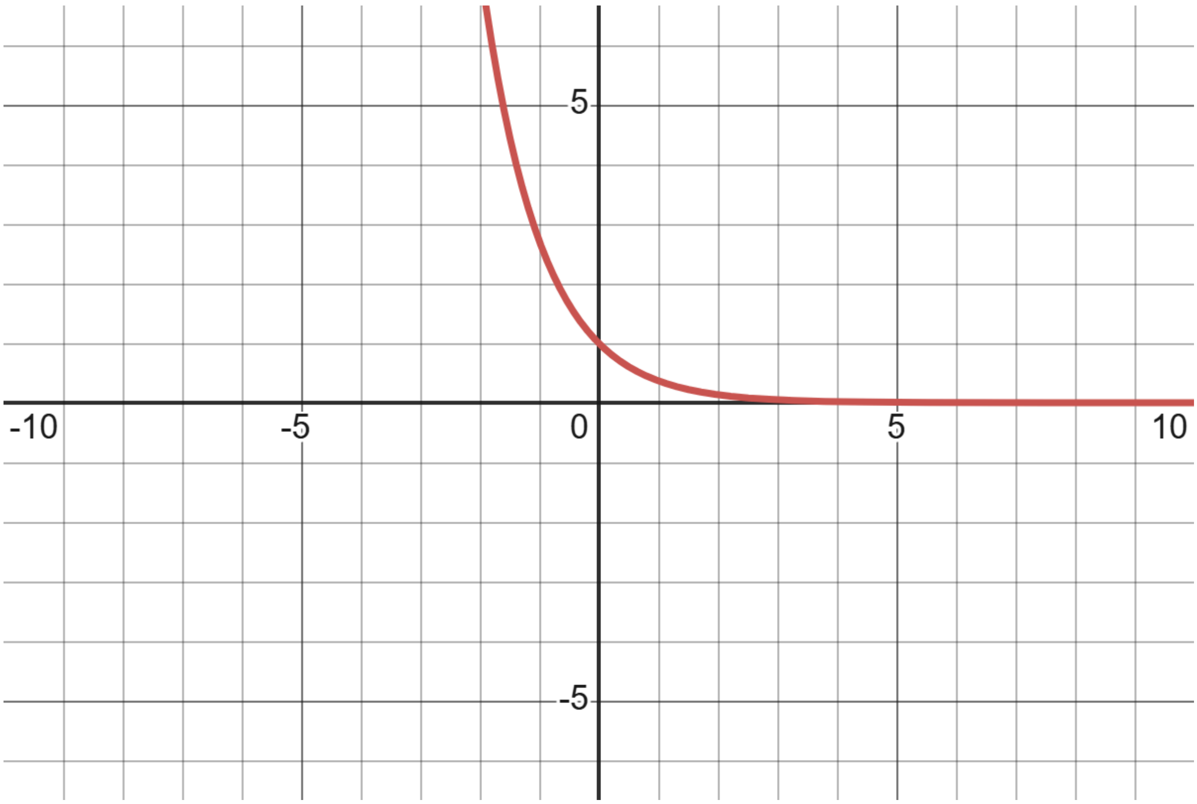


6. **Inverse Exponential**:

   For $f(x) = e^{-x}$:

$$\lim_{x \to \infty} e^{-x} = 0$$

   Since $e^{-x} = \frac{1}{e^x}$, as $x \to \infty$, $f(x)$ approaches 0.



Infinite Limits

Definition

An infinite limit refers to the situation where $f(x)$ grows without bound as x approaches a certain point a. When we write:

$$\lim_{x \to a} f(x) = \infty \quad \text{or} \quad \lim_{x \to a} f(x) = -\infty$$

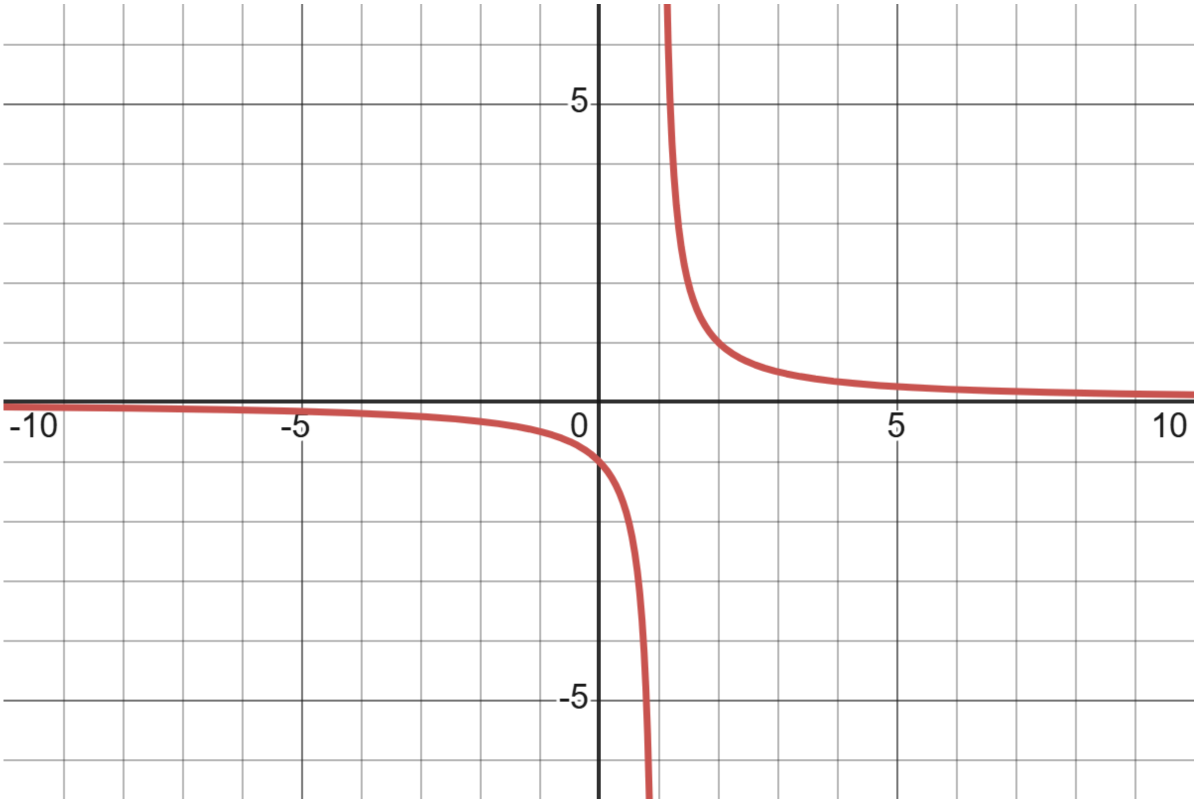
it means that $f(x)$ increases or decreases without bound as x gets close to a. This typically indicates a vertical asymptote at $x = a$.

Examples

1. **Rational function with vertical asymptote**:

   For $f(x) = \frac{1}{x-1}$, the limit as $x \to 1$ does not exist in the traditional sense because $f(x)$ grows infinitely large as it approaches 1 from the right and negatively large from the left:

$$\lim_{x \to 1^+} \frac{1}{x-1} = +\infty \quad \text{and} \quad \lim_{x \to 1^-} \frac{1}{x-1} = -\infty$$

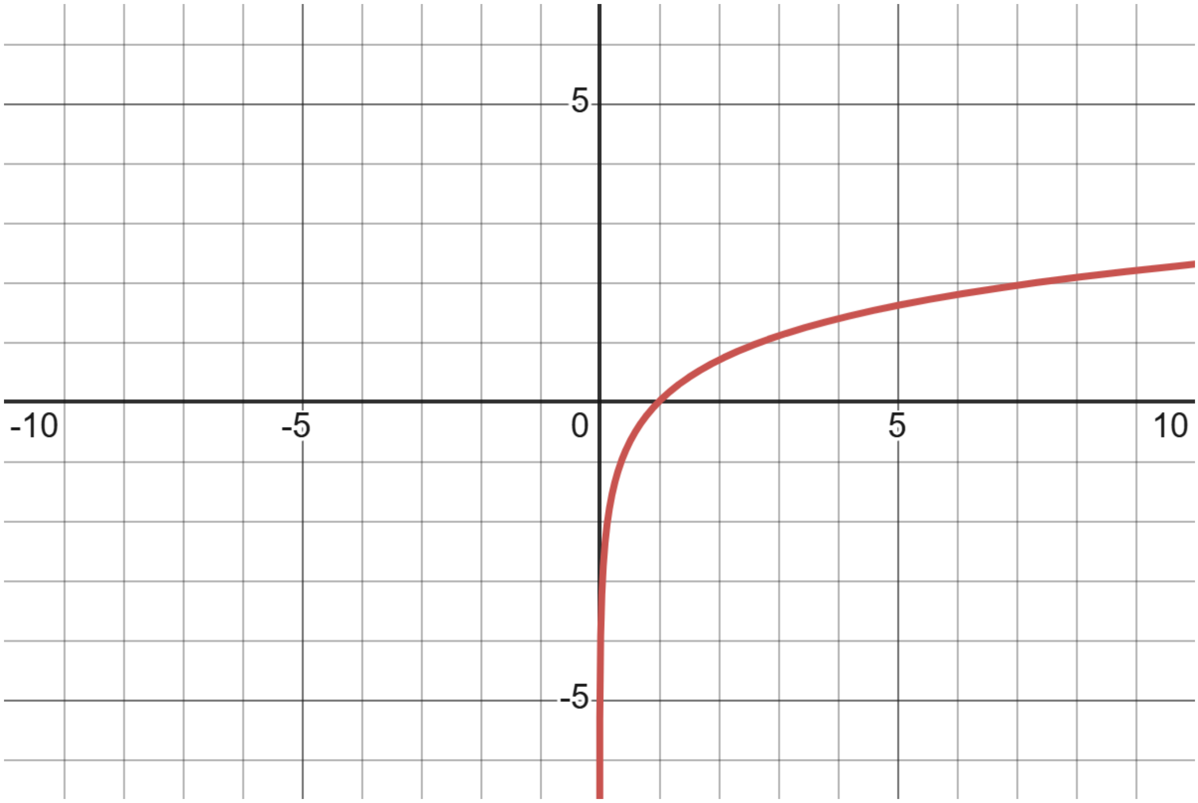


2. **Logarithmic function**:

   For $f(x) = \ln(x)$:

$$\lim_{x \to 0^+} \ln(x) = -\infty$$

As x approaches 0 from the right, $\ln(x)$ decreases without bound.

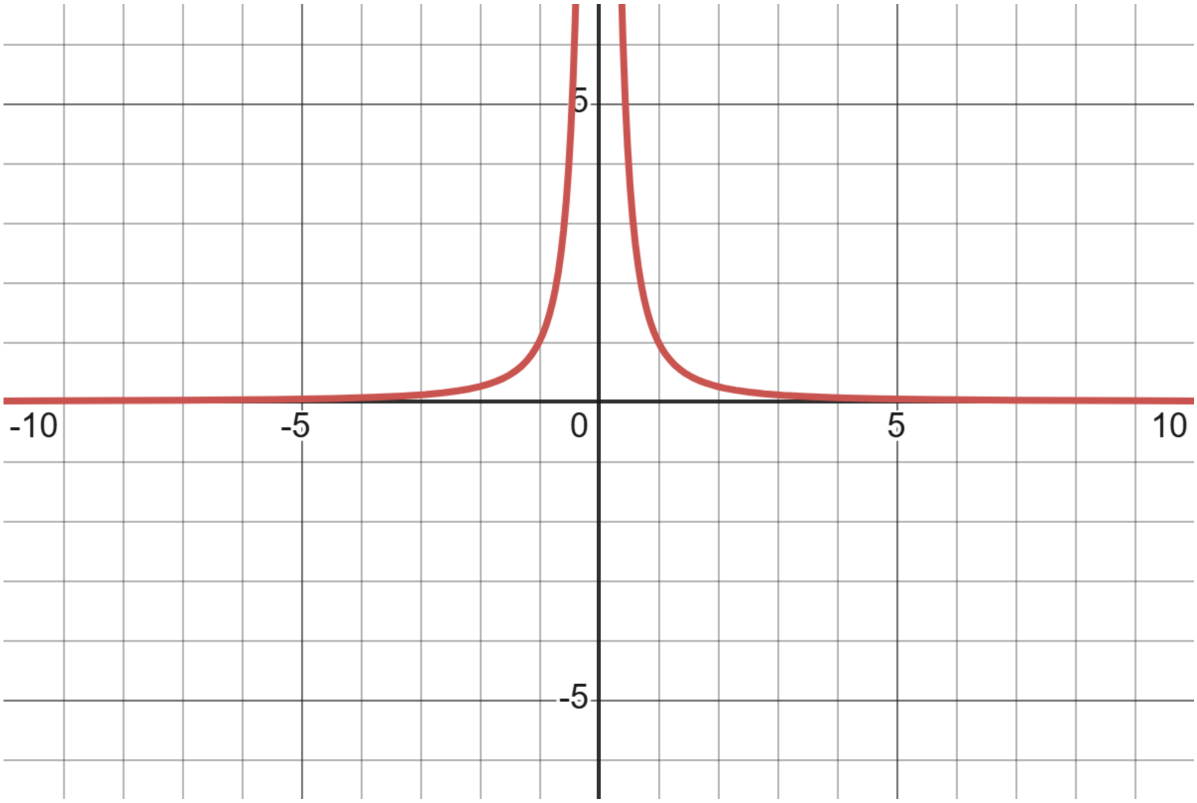


3. **Polynomial approaching zero with fractional exponent**:

   For $f(x) = \frac{1}{x^2}$, as x approaches 0:

$$\lim_{x \to 0^+} \frac{1}{x^2} = \infty \quad \text{and} \quad \lim_{x \to 0^-} \frac{1}{x^2} = \infty$$

   Regardless of approaching from the left or right, $f(x)$ goes to infinity as $x \to 0$, indicating a vertical asymptote at $x = 0$.

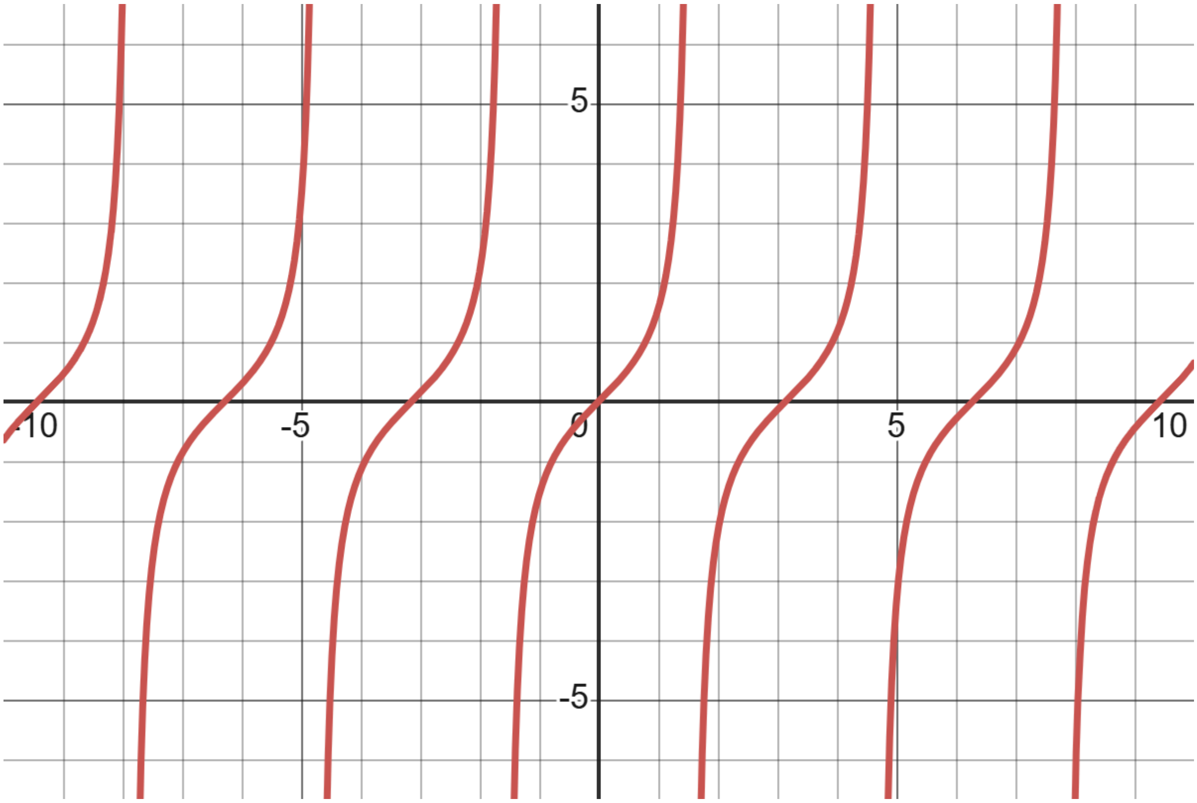


4. **Trigonometric function with infinite oscillations**:

   For $f(x) = \tan(x)$:

$$\lim_{x \to \frac{\pi}{2}^-} \tan(x) = +\infty \quad \text{and} \quad \lim_{x \to \frac{\pi}{2}^+} \tan(x) = -\infty$$

   As x approaches $\frac{\pi}{2}$ from the left, $f(x)$ goes to $+\infty$; from the right, it goes to $-\infty$.

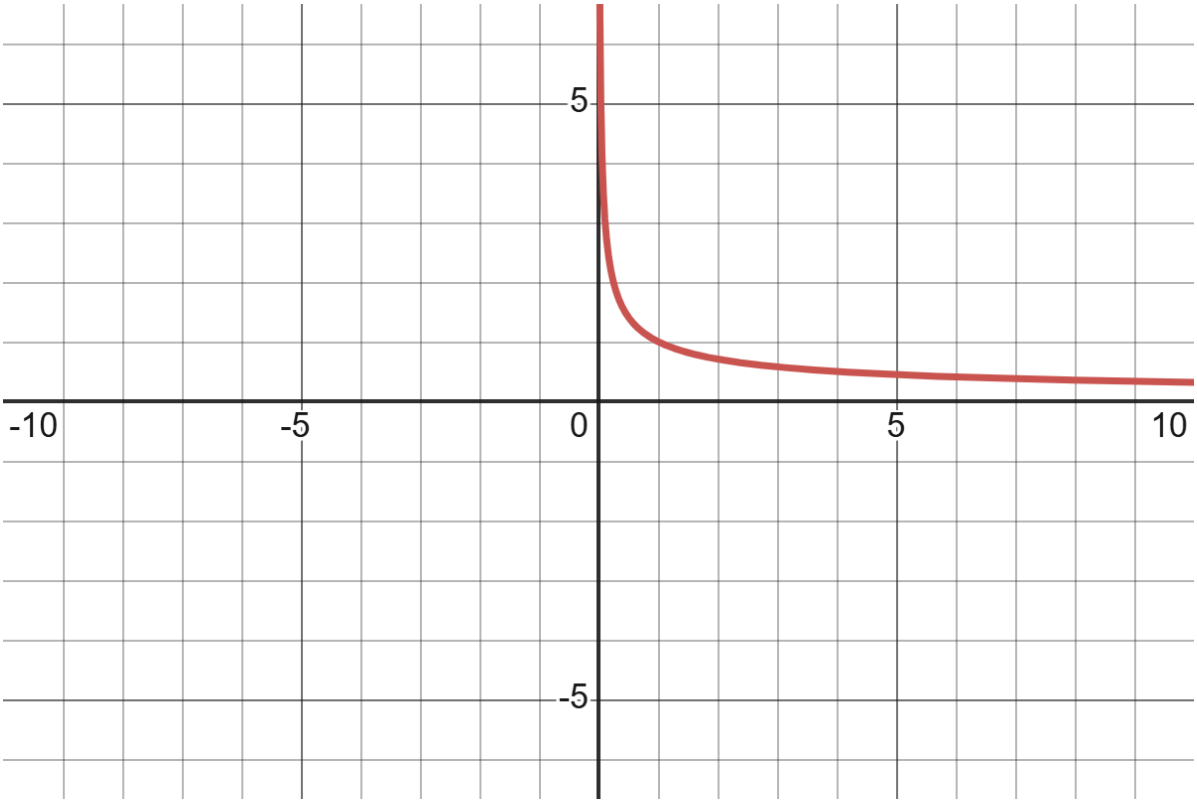


5. **Fractional power function**:

   For $f(x) = \frac{1}{\sqrt{x}}$, as $x \to 0^+$

$$\lim_{x \to 0^+} \frac{1}{\sqrt{x}} = +\infty$$

   This function approaches $+\infty$ as x gets close to 0 from the right side.



Limits at Infinity for Rational Functions

Definition

For rational functions of the form $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials, limits at infinity can be found by comparing the degrees of the polynomials.

Cases

1. **Degree of $p(x)$ < Degree of $q(x)$**:

$$\lim_{x \to \pm\infty} \frac{p(x)}{q(x)} = 0$$

2. **Degree of $p(x)$ = Degree of $q(x)$**:

$$\lim_{x \to \pm\infty} \frac{p(x)}{q(x)} = \frac{\text{leading coefficient of } p(x)}{\text{leading coefficient of } q(x)}$$

3. **Degree of $p(x)$ > Degree of $q(x)$**:

In this case, the function approaches ∞ or $-\infty$ depending on the sign.

Squeeze Theorem

Definition

The Squeeze Theorem states that if $f(x) \leq g(x) \leq h(x)$ for all x in an interval around a (excluding a itself) and:

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then $\lim_{x \to a} g(x) = L$.

Example

To evaluate $\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right)$, we can use the Squeeze Theorem. Since:

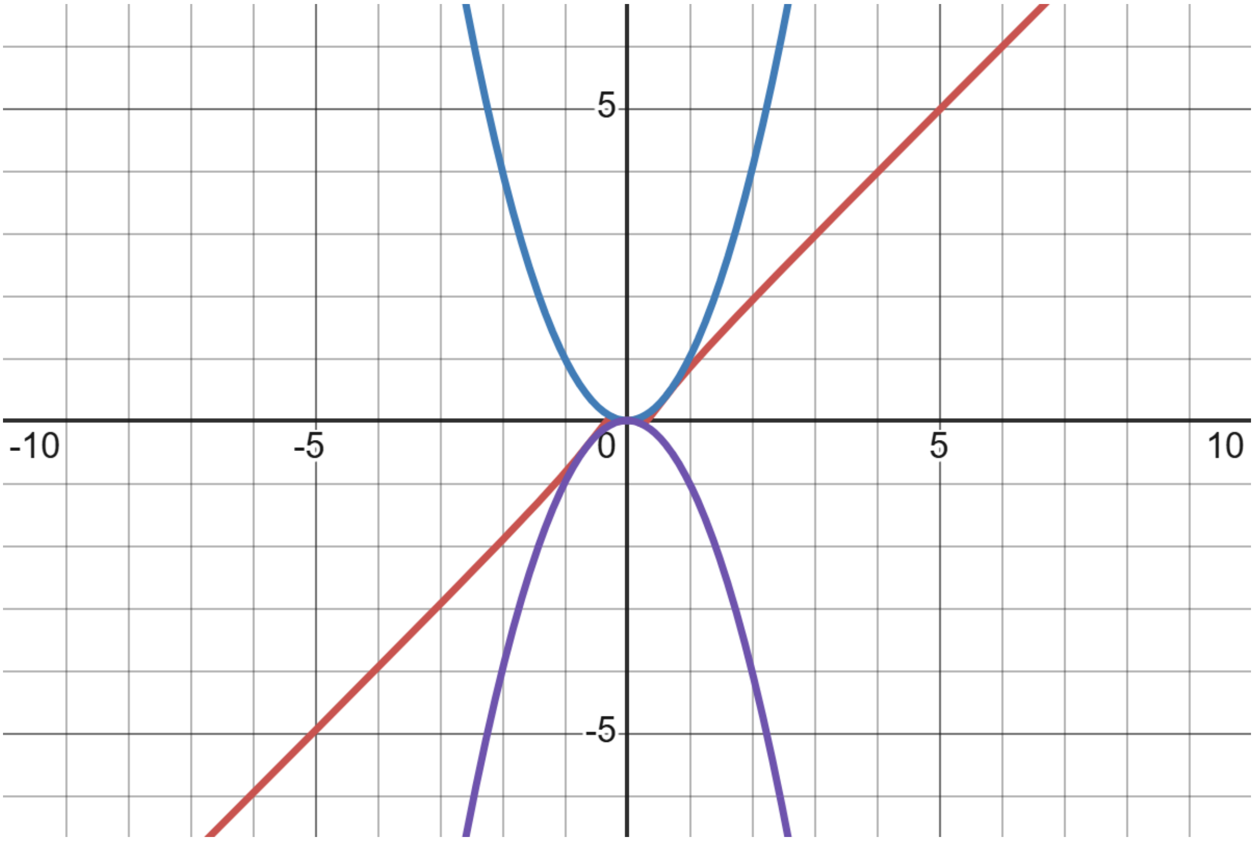
$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$$

we have:

$$-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2$$

Since both $\lim_{x \to 0} -x^2 = 0$ and $\lim_{x \to 0} x^2 = 0$, it follows from the Squeeze Theorem that:

$$\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$$



Min-Max Theorem

Definition

The **Min-Max Theorem** states that if a function $f(x)$ is **continuous** on a **closed interval** $[a, b]$, then $f(x)$ must attain both a **minimum** and a **maximum** value on that interval.

Formal Statement

If $f(x)$ is continuous on the interval $[a, b]$, then there exist points c and d in $[a, b]$ such that:

$$f(c) \leq f(x) \leq f(d) \quad \text{for all } x \in [a, b]$$

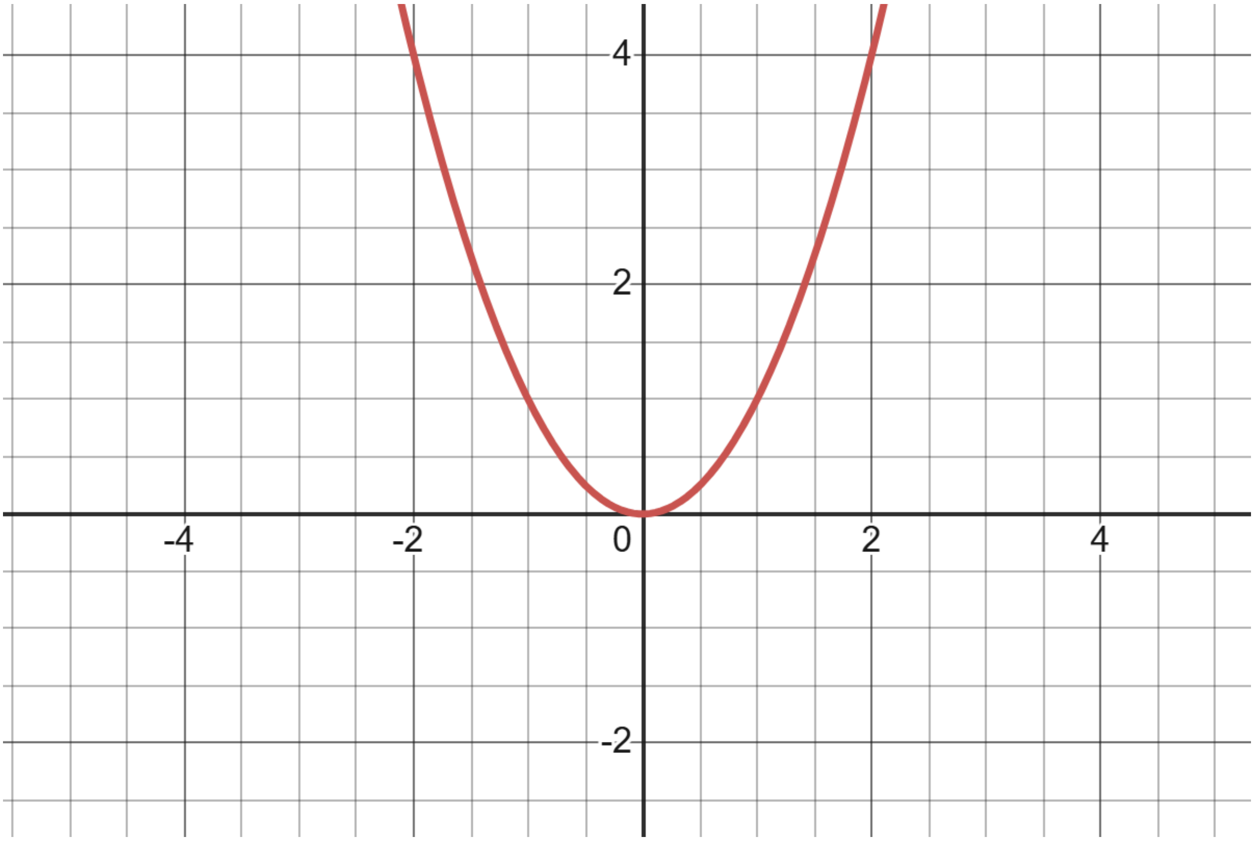
Here:

- $f(c)$ is the **minimum value** of $f(x)$ on $[a, b]$.
- $f(d)$ is the **maximum value** of $f(x)$ on $[a, b]$.

Example

Consider the function $f(x) = x^2$ on the interval $[-2, 1]$:

- The minimum value is at $x = 0$, where $f(0) = 0$.
- The maximum value is at $x = -2$, where $f(-2) = 4$.

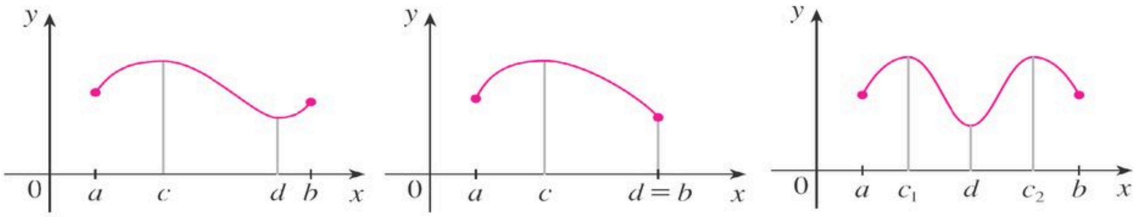


Key Points

- The theorem guarantees **absolute extremum values** (not just local extrema).
- It applies only to functions that are **continuous** on a **closed interval**.

Extreme Value Theorem: Absolute min and max

The Extreme Value Theorem is illustrated below:



Note that an extreme value can be taken on more than once.

7. Intermediate Value Theorem, Mean Value Theorem, and Derivative Basics

Intermediate Value Theorem (IVT)

Definition

The **Intermediate Value Theorem (IVT)** states that if a function $f(x)$ is **continuous** on a **closed interval** $[a, b]$ and L is any value between $f(a)$ and $f(b)$, then there exists at least one point $c \in (a, b)$ such that: $f(c) = L$

Formal Statement

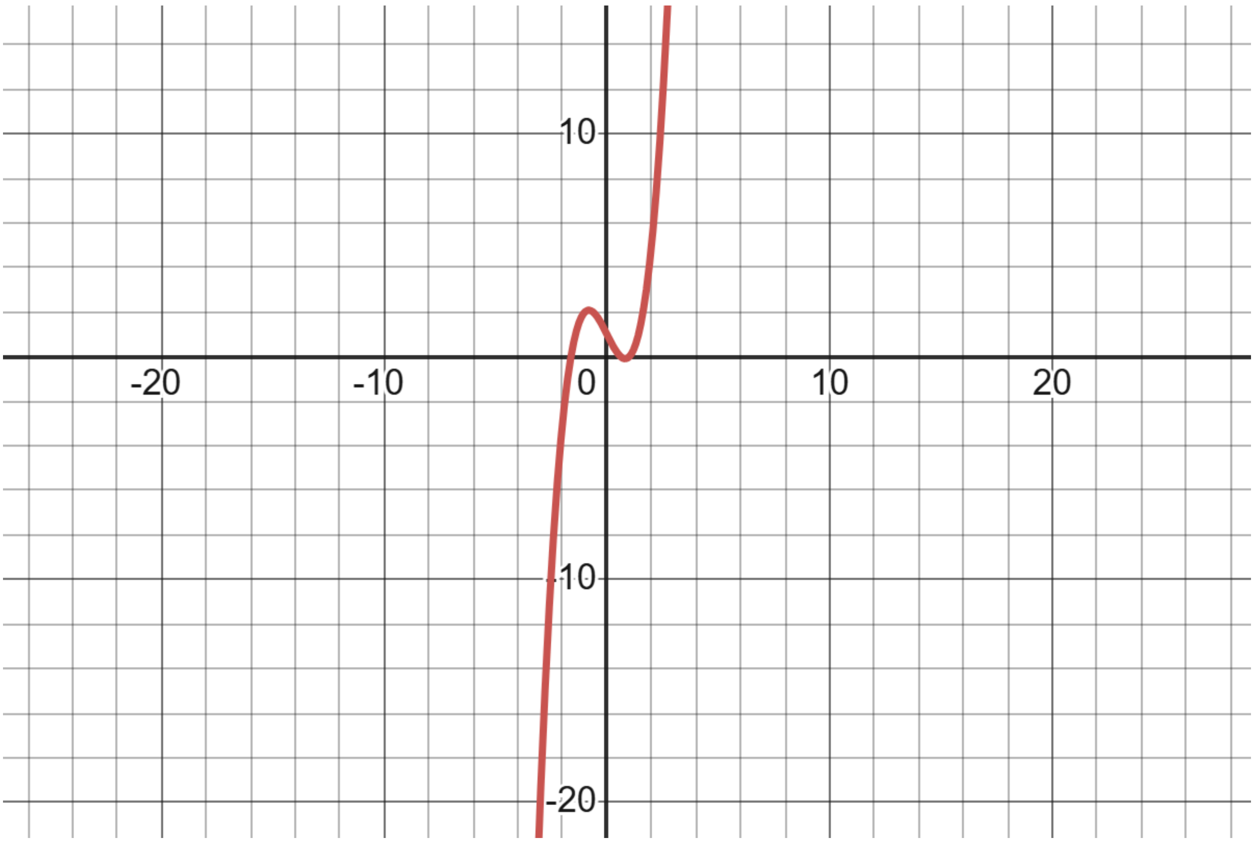
If $f(x)$ is continuous on $[a, b]$ and $f(a) \leq L \leq f(b)$ (or $f(a) \geq L \geq f(b)$), then there exists some $c \in (a, b)$ such that: $f(c) = L$

Example

If f$(x) = x^3 - 2x + 1$ and we are considering the interval $[0, 2]$:

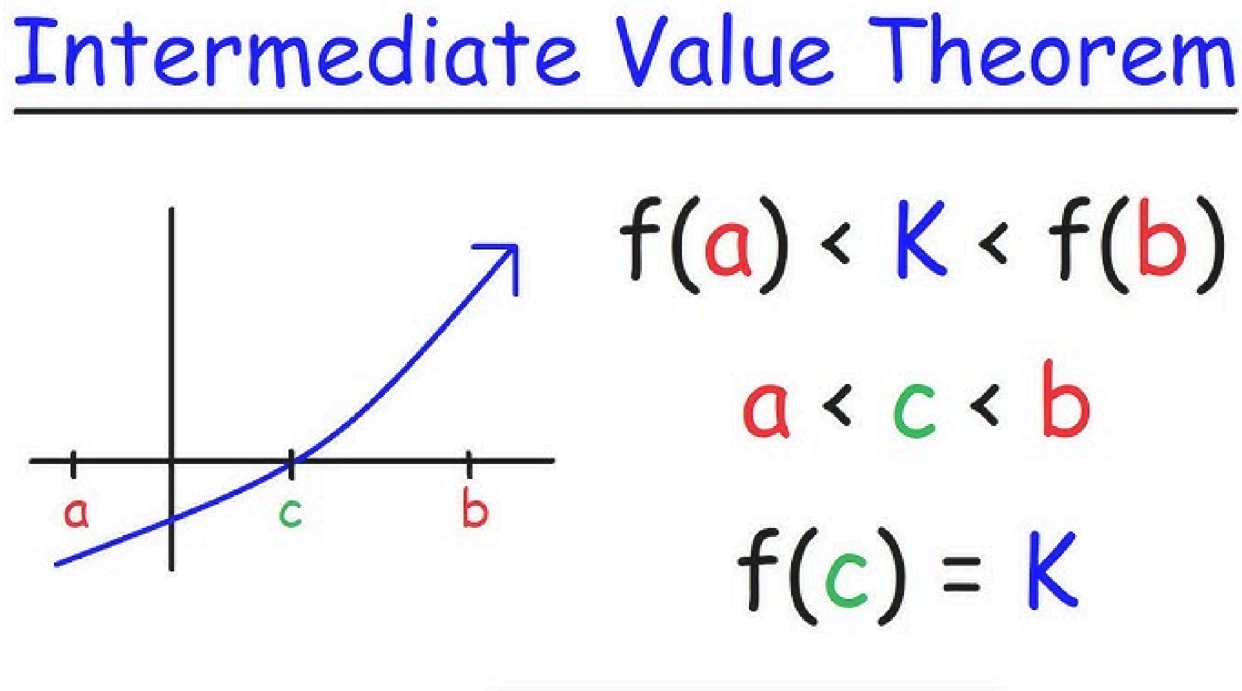
- $f(0) = 1$ and $f(2) = 5$.

- For $L = 2$, the IVT guarantees that there exists some $c \in (0, 2)$ such that $f(c) = 2$.



Key Points

- The theorem is used to **prove the existence** of solutions within an interval.
- It does **not specify** where the solution is located, only that at least one solution exists



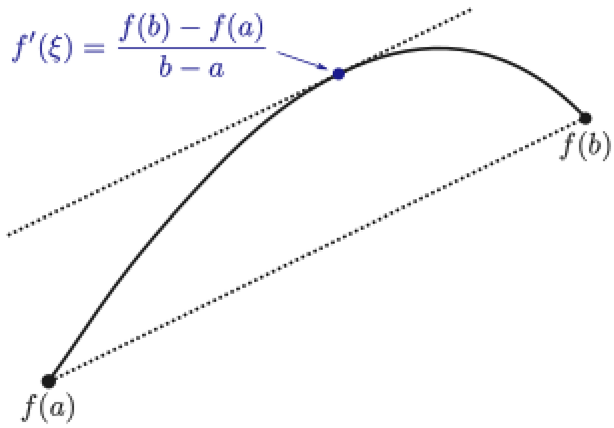
Mean Value Theorem (MVT)

Definition

The **Mean Value Theorem (MVT)** states that if a function $f(x)$ is **continuous** on a **closed interval** $[a, b]$ and **differentiable** on the **open interval** (a, b), then there exists at least one point $c \in (a, b)$ such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

This equation indicates that there is at least one point c where the **instantaneous rate of change** (the derivative) is equal to the **average rate of change** over the interval $[a, b]$.



Problem 1: Prove that $\arctan(x) - x + x^3 = -5$ has a real solution using the Intermediate Value Theorem (IVT)

To use the Intermediate Value Theorem, we need to show that the function has values of opposite signs at two points. Let's define:

$$f(x) = \arctan(x) - x + x^3 + 5$$

So, we want to show that $f(x) = 0$ has a solution.

Step 1: Choose Values for x and Calculate $f(x)$

Let's evaluate $f(x)$ at a few points to see if there is a sign change:

1. **At** $x = 0$ **:**

$$f(0) = \arctan(0) - 0 + 0^3 + 5 = 5$$

1. **At** $x = 2$ **:**

$$f(2) = \arctan(2) - 2 + 2^3 + 5 = \arctan(2) - 2 + 8 + 5$$

Since $\arctan(2) \approx 1.107$, we get:

$$f(2) \approx 1.107 - 2 + 8 + 5 = 12.107$$

1. **At** $x = -2$ **:**

$$f(-2) = \arctan(-2) - (-2) + (-2)^3 + 5 = \arctan(-2) + 2 - 8 + 5$$

Since $\arctan(-2) \approx -1.107$, we get:

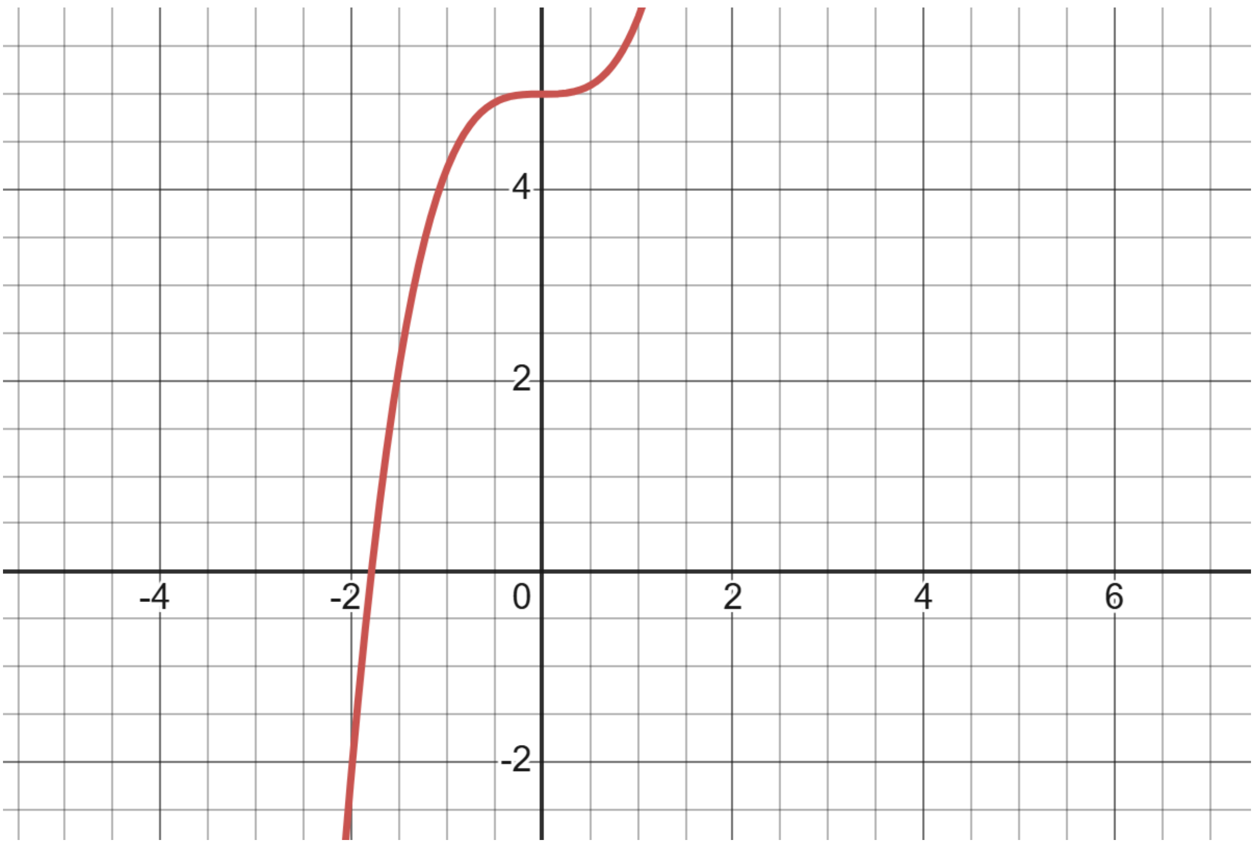
$$f(-2) \approx -1.107 + 2 - 8 + 5 = -2.107$$

Step 2: Apply the Intermediate Value Theorem

Since $f(0) = 5 > 0$ and $f(-2) \approx -2.107 < 0$, there is a sign change between $x = -2$ and $x = 0$. By the Intermediate Value Theorem, because $f(x)$ is continuous, there exists a point $x \in (-2, 0)$ where $f(x) = 0$.

Final Answer for Problem 1

Yes, the equation $\arctan(x) - x + x^3 = -5$ has a real solution in the interval $(-2, 0)$ by the Intermediate Value Theorem.



Problem 2: Prove that $f(x) = x^2 \cot(x)$ has a root using the Intermediate Value Theorem

Define $f(x) = x^2 \cot(x)$.

Step 1: Analyze the Function

Since $\cot(x) = \frac{\cos(x)}{\sin(x)}$, we have:

$$f(x) = x^2 \frac{\cos(x)}{\sin(x)}$$

The function $f(x)$ is continuous on intervals where $\sin(x) \neq 0$, i.e., where $x \neq n\pi$ for integers n.

Step 2: Choose an Interval with a Sign Change

Let's consider a small interval around $x = \pi/2$:

1. **At** $x = \pi/4$:

$$f\left(\frac{\pi}{4}\right) = \left(\frac{\pi}{4}\right)^2 \cot\left(\frac{\pi}{4}\right) = \left(\frac{\pi}{4}\right)^2 \cdot 1 = \frac{\pi^2}{16} > 0$$

1. **At** $x = 3\pi/4$:

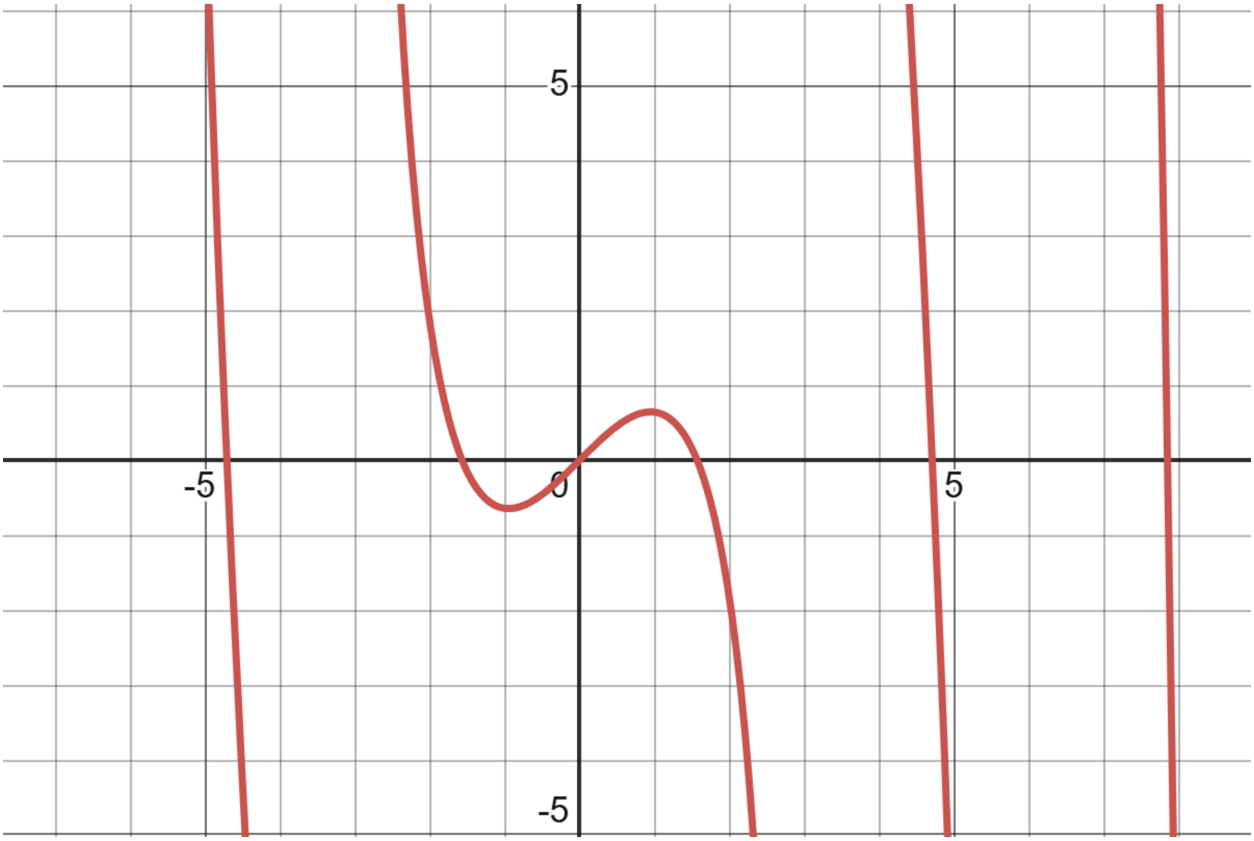
$$f\left(\frac{3\pi}{4}\right) = \left(\frac{3\pi}{4}\right)^2 \cot\left(\frac{3\pi}{4}\right) = \left(\frac{3\pi}{4}\right)^2 \cdot (-1) = -\frac{9\pi^2}{16} < 0$$

Step 3: Apply the Intermediate Value Theorem

Since $f(\pi/4) > 0$ and $f(3\pi/4) < 0$, and $f(x)$ is continuous in $(\pi/4, 3\pi/4)$, there exists a value $x \in (\pi/4, 3\pi/4)$ where $f(x) = 0$.

Final Answer for Problem 2

The function $f(x) = x^2 \cot(x)$ has a root in the interval $(\pi/4, 3\pi/4)$ by the Intermediate Value Theorem.



Tangent Lines and Derivative Definition

Tangent Lines and Secant Slope

Let $f : (a, b) \to \mathbb{R}$ be a continuous function, and let $c \in (a, b)$ be fixed. For any $x \in (a, b) \setminus \{c\}$, the **slope of the secant line** joining $(x, f(x))$ and $(c, f(c))$ is given by:

$$m(x) = \frac{f(x) - f(c)}{x - c}$$

The slope of this line gives the average rate of change of f between x and c.

Problem Point at c

As $x \to c$, the secant line's slope approaches a value (if it exists), which we define as the **derivative** of f at c.

Definition of the Derivative

The derivative of f at c, denoted by $f'(c)$, is defined by the limit:

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

if this limit exists.

Other Notations for the Derivative

- $f'(c)$
- $\frac{df}{dx}\bigg|_{x=c}$
- $Df(c)$
- $\frac{dy}{dx} if y = f(x)$

This definition captures the **instantaneous rate of change** of f at c and is the foundation of differentiation in calculus.

---

8. Differentiation and Derivatives

Tangent Lines and the Power Rule

Equation of the Tangent Line

The equation of the tangent line to the graph of f at $(a, f(a))$ is:

$$y = f(a) + f'(a)(x - a)$$

Deriving the General Power Rule

To find the derivative of $f(x) = x^n$ for any integer n, we use the limit definition of the derivative:

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$$

Using the Binomial Theorem and simplifying, we arrive at:

$$f'(x) = nx^{n-1}$$

Definition of Left and Right Derivatives

- The **left derivative** of f at a is defined as:

$$f'_-(a) = \lim x \to a^- \frac{f(x) - f(a)}{x - a}$$

- The **right derivative** of f at a is defined as:

$$f'_+(a) = \lim x \to a^+ \frac{f(x) - f(a)}{x - a}$$

Remarks on Differentiability

1. If $f'(a)$ exists, then f is called **differentiable** (or **d'able**) at $x = a$.
2. f is differentiable at $x = a$ **if and only if** both $f'_-(a)$ *and* $f'_+(a)$ exist and are equal.
3. Alternatively, setting $h = x - a$, we have:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

Example: Derivative of $f(x) = \sin(x)$

To find the derivative of $f(x) = \sin(x)$, we use the limit definition of the derivative. For a function $f(x)$, the derivative at any point $x = a$ is given by

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

For $f(x) = \sin(x)$, we want to compute $f'(a)$.

Step-by-Step Solution

1. **Set Up the Derivative Using the Definition**:
   Substitute $f(x) = \sin(x)$:

$$f'(a) = \lim_{h \to 0} \frac{\sin(a + h) - \sin(a)}{h}$$

2. **Apply the Sine Addition Formula**:
   Using $\sin(a + h) = \sin(a)\cos(h) + \cos(a)\sin(h)$, we get:

$$f'(a) = \lim_{h \to 0} \frac{\sin(a)\cos(h) + \cos(a)\sin(h) - \sin(a)}{h}$$

3. **Factor Out $\sin(a)$ and $\cos(a)$**:
   Rewrite the expression by grouping terms with $\sin(a)$ and $\cos(a)$:

$$f'(a) = \lim_{h \to 0} \left(\sin(a)\frac{\cos(h) - 1}{h} + \cos(a)\frac{\sin(h)}{h} \right)$$

4. **Evaluate Each Limit**:
   - It is known that $\lim_{h \to 0} \frac{\sin(h)}{h} = 1$.
   - It is also known that $\lim_{h \to 0} \frac{\cos(h)-1}{h} = 0$.

   Substituting these limits, we get:

$$f'(a) = \sin(a) \cdot 0 + \cos(a) \cdot 1 = \cos(a)$$

Final Result

Thus, the derivative of $f(x) = \sin(x)$ is

$$f'(x) = \cos(x)$$

Algebra of Derivatives

Let f and g be two functions for which $f'(a)$ and $g'(a)$ exist. Then:

1. **Sum and Difference Rules**:

$$(f+g)'(x) = f'(x) + g'(x), \quad (f-g)'(x) = f'(x) - g'(x)$$

2. **Product Rule (Leibniz Rule)**:

$$(f \cdot g)'(x) = f'(x)g(x) + f(x)g'(x)$$

3. **Quotient Rule**:

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2} \quad \text{for } g(x) \neq 0$$

4. **Reciprocal Rule**: If $f(x) \neq 0$, then the derivative of $\frac{1}{f(x)}$ is given by:

$$\left(\frac{1}{f}\right)'(x) = -\frac{f'(x)}{f(x)^2}$$

Examples of Derivatives

1. $f(x) = x^4 + \sin(x)$

$$f'(x) = 4x^3 + \cos(x)$$

1. $f(x) = x \cdot \sin(x)$

$$f'(x) = \sin(x) + x \cdot \cos(x)$$

1. $f(x) = \tan(x) = \frac{\sin(x)}{\cos(x)}$

$$f'(x) = \frac{\cos(x) \cdot \cos(x) - \sin(x) \cdot (-\sin(x))}{\cos(x)^2} = 1 + \tan^2(x) = \sec^2(x)$$

Chain Rule and Applications

Chain Rule for Derivatives

For a composite function $h(x) = f(g(x))$, the derivative is given by:

$$h'(x) = f'(g(x)) \cdot g'(x)$$

Example: $h(x) = 1 + \sin^2(x)$

- Let $f(x) = 1 + x^2$ and $g(x) = \sin(x)$. Then:

$$h(x) = f(g(x)) = 1 + \sin^2(x)$$

- By the Chain Rule:

$$h'(x) = 2\sin(x) \cdot \cos(x) = \sin(2x)$$

where we used the double-angle identity.

Another Example: $h(x) = \sin(x^2 + 1) + 1$

To find the derivative of $h(x) = \sin(x^2 + 1) + 1$ using the Chain Rule, let's go through the steps:

1. **Identify the Outer and Inner Functions**:
   - The outer function is $f(u) = \sin(u) + 1$.
   - The inner function is $g(x) = x^2 + 1$.
2. **Apply the Chain Rule**:

   According to the Chain Rule, $h'(x) = f'(g(x)) \cdot g'(x)$.
3. **Differentiate the Outer Function**:

   The derivative of $f(u) = \sin(u) + 1$ with respect to u is $f'(u) = \cos(u)$.
4. **Differentiate the Inner Function**:

   The derivative of $g(x) = x^2 + 1$ with respect to x is $g'(x) = 2x$.
5. **Combine Results**:

   Substitute $g(x) = x^2 + 1$ and $g'(x) = 2x$ into the Chain Rule formula:

$$h'(x) = f'(g(x)) \cdot g'(x) = \cos(x^2 + 1) \cdot 2x$$

Final Result

Thus, the derivative of $h(x) = \sin(x^2 + 1) + 1$ is

$$h'(x) = \cos(x^2 + 1) \cdot 2x$$

Remark on the Derivative of Inverse Functions

If f is differentiable in a neighborhood of a point a, and if f is one-to-one and onto in this neighborhood, then the composite $f(f^{-1}(x)) = x$. By using the Chain Rule, we can differentiate this composition to find the derivative of the inverse function.

Derivation of the Formula for the Derivative of an Inverse Function

Suppose $f(x)$ is a function that is differentiable and has an inverse $f^{-1}(x)$. We want to find the derivative of $f^{-1}(x)$ at a point x, which is given by the formula:

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

provided that $f'(f^{-1}(x)) \neq 0$.

Step-by-Step Derivation

1. **Express the Inverse Relationship**:
   Since f and f^{-1} are inverses, for any point x in the domain of f^{-1}, we have:

   $$f(f^{-1}(x)) = x$$

2. **Differentiate Both Sides with Respect to x**:
   Differentiate the equation $f(f^{-1}(x)) = x$ with respect to x. Using the Chain Rule on the left side, we get:

   $$\frac{d}{dx}\left(f(f^{-1}(x))\right) = \frac{d}{dx}(x)$$

3. **Apply the Chain Rule**:
   By the Chain Rule, the derivative of $f(f^{-1}(x))$ is $f'(f^{-1}(x)) \cdot (f^{-1})'(x)$. On the right side, the derivative of x is 1. So we have:

   $$f'(f^{-1}(x)) \cdot (f^{-1})'(x) = 1$$

4. **Solve for $(f^{-1})'(x)$:**
   To isolate $(f^{-1})'(x)$, divide both sides by $f'(f^{-1}(x))$:

   $$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

Conclusion

Thus, we have derived the formula for the derivative of an inverse function:

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

This formula is valid as long as $f'(f^{-1}(x)) \neq 0$, ensuring the denominator is non-zero.

Example: Finding the Derivative of $f(x) = \tan(x)$ and Its Inverse $f^{-1}(x) = \arctan(x)$

Part 1: Derivative of $f(x) = \tan(x)$

1. **Function Definition**:

   For $f(x) = \tan(x)$, we want to find $f'(x)$, the derivative of $\tan(x)$ with respect to x.

2. **Differentiate $\tan(x)$**:

   We know that the derivative of $\tan(x)$ is given by:

   $$f'(x) = \sec^2(x)$$

   So, the derivative of $f(x) = \tan(x)$ is:

   $$f'(x) = \sec^2(x)$$

Part 2: Derivative of the Inverse Function $f^{-1}(x) = \arctan(x)$

Now, we'll find the derivative of the inverse function $f^{-1}(x) = \arctan(x)$ by using implicit differentiation.

1. **Relationship with the Tangent Function**:

   Since $\arctan(x)$ is the inverse of $\tan(x)$, we have:

   $$y = \arctan(x) \Rightarrow x = \tan(y)$$

2. **Implicit Differentiation**:

   Rewrite the relationship as $x = \tan(y)$ and differentiate both sides with respect to x:

$$\frac{d}{dx}(x) = \frac{d}{dx}(\tan(y))$$

Since the derivative of x with respect to x is 1, we get:

$$1 = \sec^2(y) \cdot \frac{dy}{dx}$$

3. **Solve for $\frac{dy}{dx}$:**

   To find $\frac{dy}{dx}$, isolate it by dividing both sides by $\sec^2(y)$:

$$\frac{dy}{dx} = \frac{1}{\sec^2(y)}$$

4. **Express $\sec^2(y)$ in Terms of x :**

   Since $x = \tan(y)$, we can use the identity $\sec^2(y) = 1 + \tan^2(y)$ to express $\sec^2(y)$ in terms of x :

$$\sec^2(y) = 1 + x^2$$

5. **Substitute and Simplify**:

   Substitute $\sec^2(y) = 1 + x^2$ back into the equation for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{1}{1 + x^2}$$

Final Answer

The derivative of $f^{-1}(x) = \arctan(x)$ is:

$$\frac{d}{dx}\arctan(x) = \frac{1}{1 + x^2}$$

Homework Exercises

1. **HW1**: If $f(x) = \cos(x)$, find $f'(x)$. Prove these rules using the limit definition of the derivative.
2. **HW2**: Compute the derivatives of all inverse trigonometric functions.

HW1: If $f(x) = \cos(x)$, Find $f'(x)$. Prove Using the Limit Definition of the Derivative

To find $f'(x) for f(x) = \cos(x)$ using the limit definition of the derivative, we start with

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

For $f(x) = \cos(x)$, this becomes

$$f'(x) = \lim_{h \to 0} \frac{\cos(x+h) - \cos(x)}{h}$$

Using the cosine addition formula $\cos(x+h) = \cos(x)\cos(h) - \sin(x)\sin(h)$, substitute this into the limit:

$$f'(x) = \lim_{h \to 0} \frac{\cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x)}{h}$$

Factor out $\cos(x)$ from the first and last terms in the numerator:

$$f'(x) = \lim_{h \to 0} \frac{\cos(x)(\cos(h) - 1) - \sin(x)\sin(h)}{h}$$

Separate the terms in the numerator:

$$f'(x) = \lim_{h \to 0} \left(\cos(x) \cdot \frac{\cos(h) - 1}{h} - \sin(x) \cdot \frac{\sin(h)}{h} \right)$$

Now, we can use two well-known trigonometric limits:

1. $\lim_{h \to 0} \frac{\sin(h)}{h} = 1$
2. $\lim_{h \to 0} \frac{\cos(h) - 1}{h} = 0$

Substituting these values gives:

$$f'(x) = \cos(x) \cdot 0 - \sin(x) \cdot 1$$

$$f'(x) = -\sin(x)$$

Therefore, the derivative of $f(x) = \cos(x)$ is

$$f'(x) = -\sin(x)$$

HW2: Compute the Derivatives of All Inverse Trigonometric Functions

Here we will find the derivatives of each inverse trigonometric function. We'll use the fact that if $y = f^{-1}(x)$, then $\frac{dy}{dx} = \frac{1}{f'(f^{-1}(x))}$.

1. Derivative of $\arcsin(x)$

Let $y = \arcsin(x)$. Then $x = \sin(y)$.

Using the identity $\cos^2(y) = 1 - \sin^2(y)$, we get $\cos(y) = \sqrt{1 - x^2}$.

Since $\frac{d}{dy}\sin(y) = \cos(y)$, we have:

$$\frac{dx}{dy} = \cos(y) = \sqrt{1 - x^2}$$

Thus,

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}$$

So,

$$\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1 - x^2}}$$

2. Derivative of $\arccos(x)$

Let $y = \arccos(x)$. Then $x = \cos(y)$.

Differentiating with respect to y :

$$\frac{dx}{dy} = -\sin(y) = -\sqrt{1 - x^2}$$

Thus,

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1 - x^2}}$$

So,

$$\frac{d}{dx} \arccos(x) = -\frac{1}{\sqrt{1 - x^2}}$$

3. Derivative of $\arctan(x)$

Let $y = \arctan(x)$. Then $x = \tan(y)$.

Using $\sec^2(y) = 1 + \tan^2(y)$, we have $\sec^2(y) = 1 + x^2$.

Thus,

$$\frac{dy}{dx} = \frac{1}{1 + x^2}$$

So,

$$\frac{d}{dx} \arctan(x) = \frac{1}{1 + x^2}$$

4. Derivative of $\cot^{-1}(x)$

Let $y = \cot^{-1}(x)$. Then $x = \cot(y)$.

Using $\csc^2(y) = 1 + \cot^2(y)$, we have $\csc^2(y) = 1 + x^2$.

Thus,

$$\frac{dy}{dx} = -\frac{1}{1 + x^2}$$

So,

$$\frac{d}{dx} \cot^{-1}(x) = -\frac{1}{1 + x^2}$$

5. Derivative of $\sec^{-1}(x)$

Let $y = \sec^{-1}(x)$. Then $x = \sec(y)$.

Using $\sec^2(y) - 1 = \tan^2(y)$, we get $|\tan(y)| = \sqrt{x^2 - 1}$.

Thus,

$$\frac{dy}{dx} = \frac{1}{|x|\sqrt{x^2 - 1}}$$

So,

$$\frac{d}{dx}\sec^{-1}(x) = \frac{1}{|x|\sqrt{x^2 - 1}}$$

6. Derivative of $\csc^{-1}(x)$

Let $y = \csc^{-1}(x)$. Then $x = \csc(y)$.

Using $\csc^2(y) - 1 = \cot^2(y)$, we get $|\cot(y)| = \sqrt{x^2 - 1}$.

Thus,

$$\frac{dy}{dx} = -\frac{1}{|x|\sqrt{x^2 - 1}}$$

So,

$$\frac{d}{dx}\csc^{-1}(x) = -\frac{1}{|x|\sqrt{x^2 - 1}}$$

Derivatives of Inverse Trigonometric Functions

1. $\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}$
2. $\frac{d}{dx}\arccos(x) = -\frac{1}{\sqrt{1-x^2}}$
3. $\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$
4. $\frac{d}{dx}\cot^{-1}(x) = -\frac{1}{1+x^2}$

5. $\frac{d}{dx} \sec^{-1}(x) = \frac{1}{|x|\sqrt{x^2-1}}$
6. $\frac{d}{dx} \csc^{-1}(x) = -\frac{1}{|x|\sqrt{x^2-1}}$

9. Implicit Differentiation and Tangent Lines

Implicit Differentiation

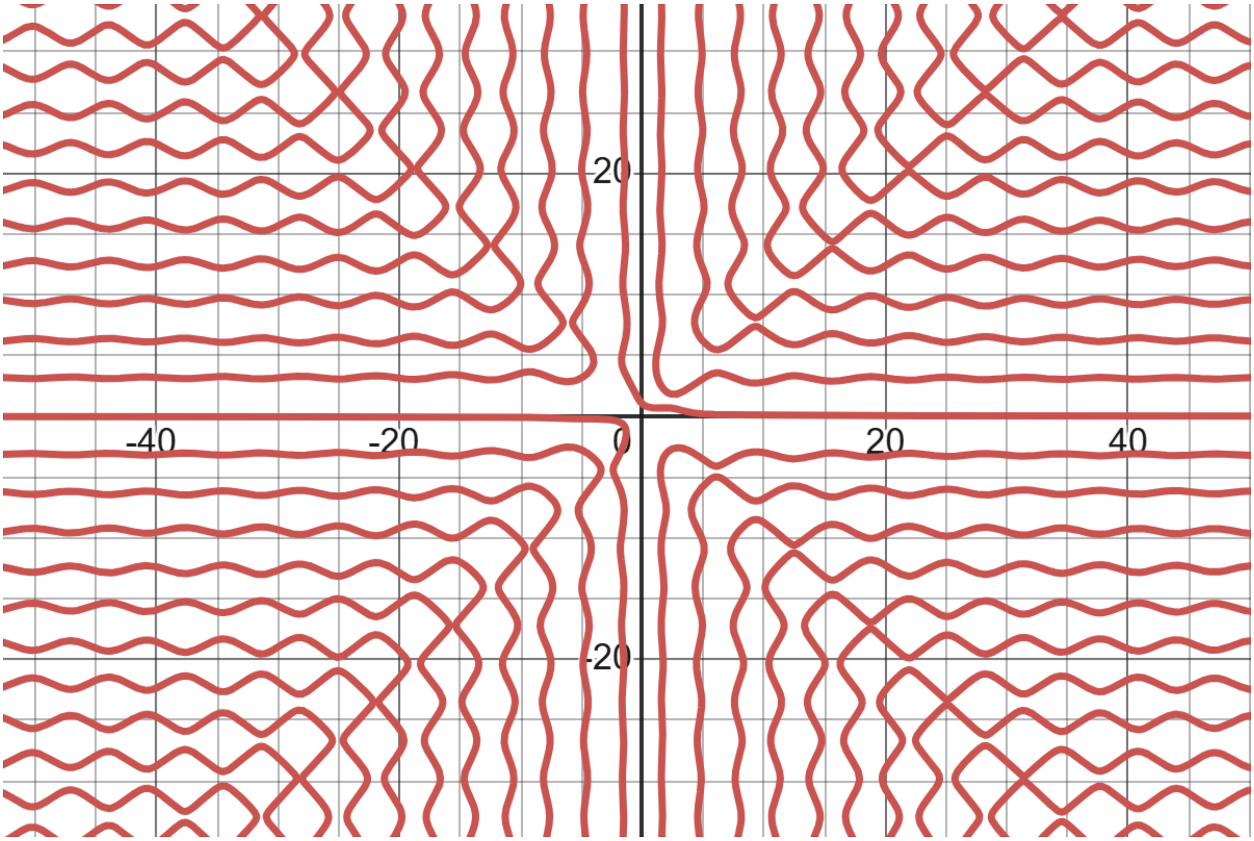
Often, the relationship between variables x and y is given by an equation rather than a function. This requires **implicit differentiation**.

Example

Let $F(x, y) = x \sin(y) + y \cos(x) - 1.$

Geometrically, this relation corresponds to the curve:

$$C := \{(x, y) \in \mathbb{R}^2 \mid F(x, y) = 0\}$$



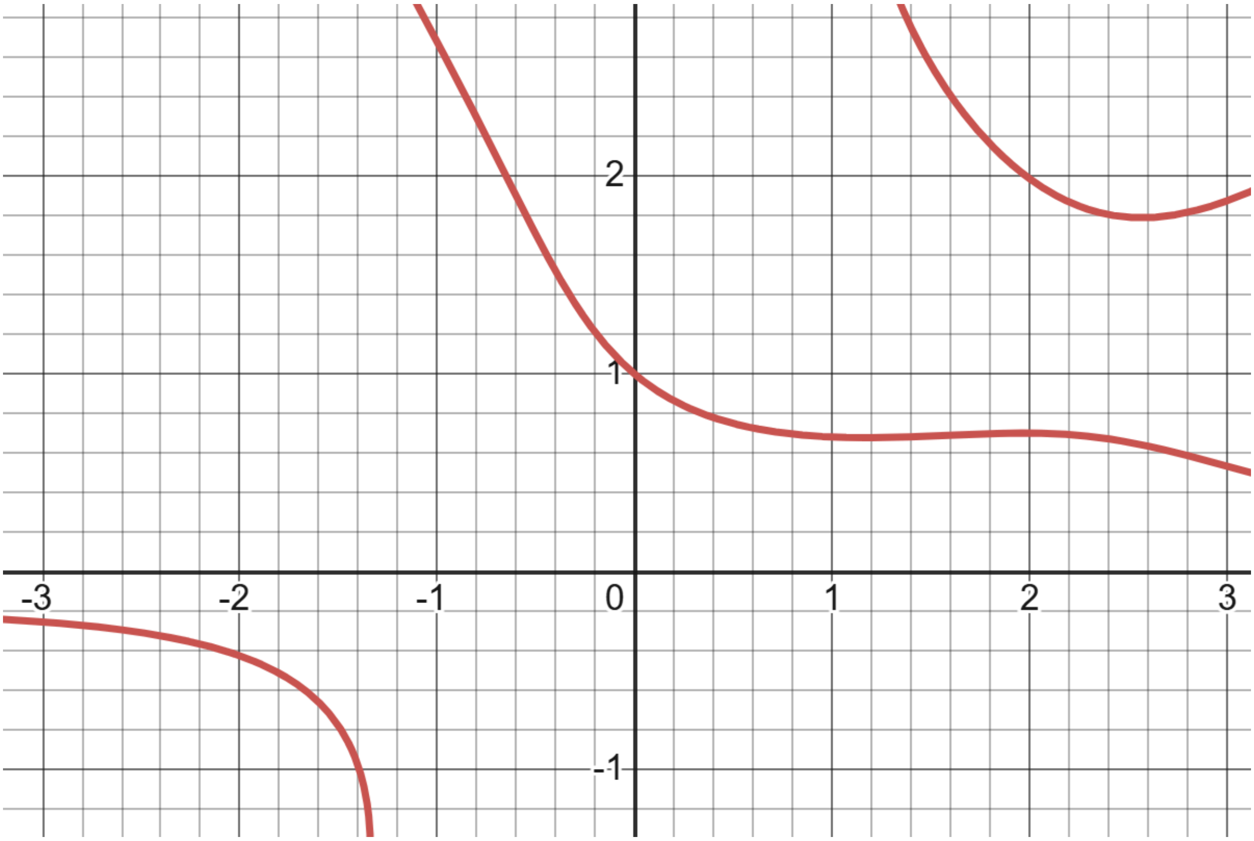
Verifying a Point on the Curve

Let $(\alpha, \beta) = (0, 1)$. Substitute $x = 0$ and $y = 1$ into $F(x, y)$:

$$F(0,1) = 0 \cdot \sin(1) + 1 \cdot \cos(0) - 1 = 1 - 1 = 0$$

Thus, $(0, 1)$ is on the curve $F(x, y) = 0$.

Assume that around $(\alpha, \beta) = (0, 1)$, y can be expressed as a function of $x : y = y(x)$.



Differentiating Implicitly

Differentiate both sides of $F(x, y(x)) = 0$ with respect to x:

$$\sin(y) + x\cos(y(x)) \cdot y'(x) + y'(x)\cos(x) + y(x) \cdot (-\sin(x)) = 0$$

Setting $(x, y) = (0, 1)$

Substitute (x, y) = (0, 1) :

$$1 \cdot \sin(1) + 0 \cdot \cos(1) \cdot y'(0) + y'(0) \cdot \cos(0) + 1 \cdot (-\sin(0)) = 0$$

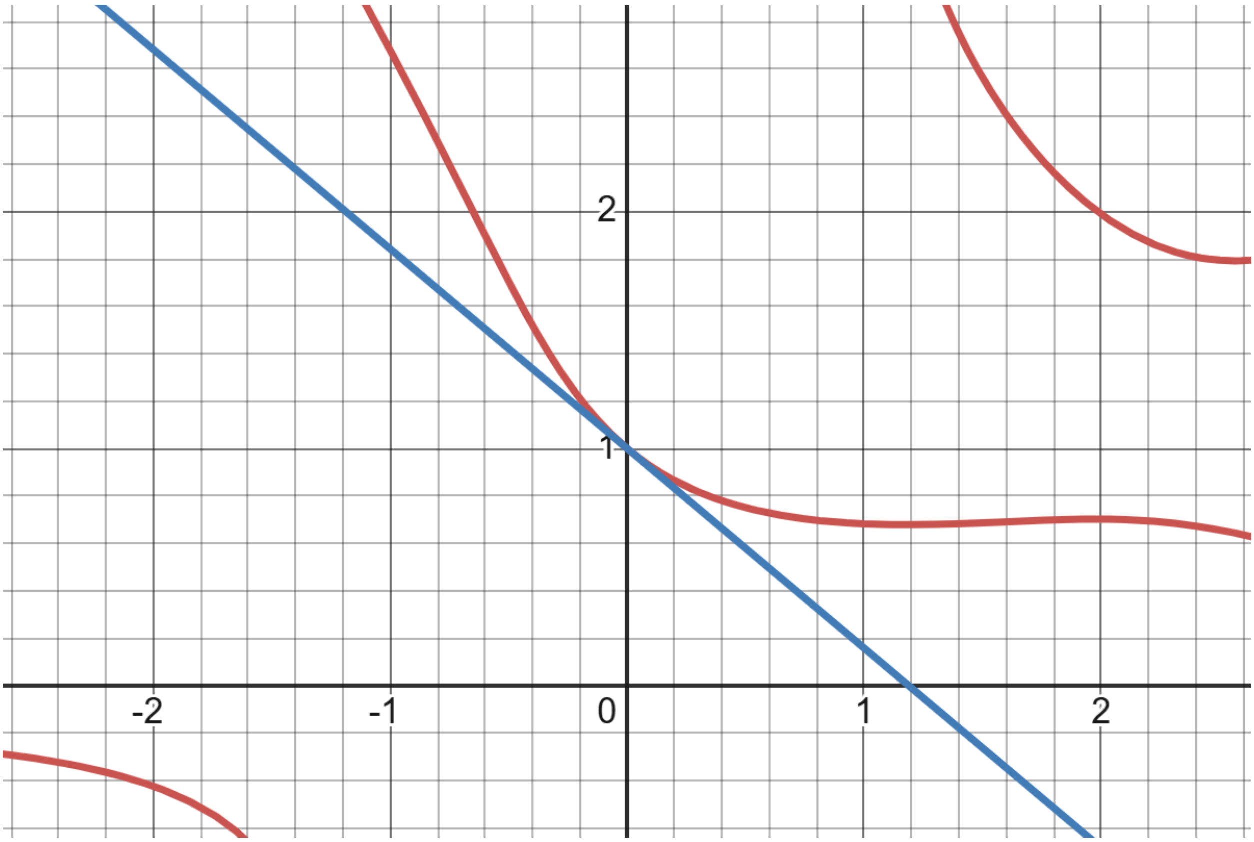
This simplifies to:

$$y'(0) = -\sin(1)$$

Equation of the Tangent Line

Using the point-slope form, the tangent line at (0, 1) is:

$$(y-1) = y'(0)(x-0) \Rightarrow y = -\sin(1) \cdot x + 1$$



Alternative Method: Assuming $x = x(y)$

Similarly, if we assume x can be expressed as a function of y, we differentiate with respect to y.

Remark on Expressibility

Around a point (α, β) :

- If $y'(\alpha) = 0$, then x cannot generally be expressed as a function of y locally.
- If $x'(\delta) = 0$, then y cannot generally be expressed as a function of x locally.
- Exceptions to these general cases do exist.

Theorem: Differentiability Implies Continuity

Let $f : (b, c) \to \mathbb{R}$ be a function, and let $a \in (b, c)$. If $f'(a)$ exists (i.e., f is differentiable at a), then:

$$\lim_{x \to a} f(x) = f(a)$$

This means that f is **continuous** at $x = a$. In particular, **continuity is a necessary condition** for differentiability.

Example Problem

Let

$$f(x) = \begin{cases} \sin(x) + 1 & \text{if } x \geq 0 \\ ax + b & \text{if } x < 0 \end{cases}$$

Given that f is differentiable on $\mathbb{R}$, find a and b.

Solution

1. **Continuity at $x = 0$:**

   $$\lim_{x \to 0^+} f(x) = \sin(0) + 1 = 1$$

   $$= \lim_{x \to 0^-} f(x) = a \cdot 0 + b = b$$

   For continuity, $b = 1$.

2. **Differentiability at $x = 0$:**

   $$f'(x) = \begin{cases} \cos(x) & \text{if } x > 0 \\ a & \text{if } x < 0 \end{cases}$$

   We calculate the derivative at $x = 0$ from both sides:

   $$\lim_{x \to 0^+} f'(x) = \cos(0) = 1$$

$$\lim_{x \to 0^-} f'(x) = a$$

For differentiability, $a = 1$.

Final Answer

$a = 1$ and $b = 1$.

10. Higher Order Derivatives and the Mean Value Theorem (MVT)

Higher Order Derivatives

The **higher order derivatives** of a function $f(x)$ are obtained by repeatedly differentiating $f(x)$.

1. **First Derivative**: $f'(x)$ (rate of change or slope of $f(x)$)
2. **Second Derivative**: $f''(x)$, obtained by differentiating $f'(x)$. This represents the rate of change of the slope (e.g., concavity of $f(x)$).
3. **Third Derivative**: $f'''(x)$, obtained by differentiating $f''(x)$.
4. n**-th Derivative**: Denoted $f^{(n)}(x)$, obtained by differentiating $f(x)$ exactly n times.

Example: For $f(x) = x^3$:

- $f'(x) = 3x^2$
- $f''(x) = 6x$
- $f'''(x) = 6$
- $f^{(n)}(x) = 0$ for $n \geq 4$ (constant derivative).

The Mean Value Theorem (MVT)

Statement of the Theorem

Let $f : [a, b] \to \mathbb{R}$ be a function. If:

1. f is **continuous** on $[a, b]$,
2. f is **differentiable** on (a, b),

then there exists some $c \in (a, b)$ such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Intuition

The Mean Value Theorem states that there is at least one point c in (a, b) where the instantaneous rate of change (the derivative) equals the average rate of change over the interval.

Remark on Monotonicity and Derivatives

Let $f : [a, b] \to \mathbb{R}$ be a function. We define:

1. f is **increasing** on $[a, b]$ if for all $x_1, x_2 \in [a, b]$ with $x_1 < x_2$, $f(x_1) < f(x_2)$.
2. f is **decreasing** on $[a, b]$ if for all $x_1, x_2 \in [a, b]$ with $x_1 < x_2$, $f(x_1) > f(x_2)$.
3. f is **non-decreasing** on $[a, b]$ if for all $x_1, x_2 \in [a, b]$ with $x_1 < x_2$, $f(x_1) \leq f(x_2)$.
4. f is **non-increasing** on $[a, b]$ if for all $x_1, x_2 \in [a, b]$ with $x_1 < x_2$, $f(x_1) \geq f(x_2)$.

If f is continuous on $[a, b]$ and differentiable on (a, b), then for any $x_1, x_2 \in [a, b]$, the **MVT** ensures that:

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(x) \quad \text{for some } x \in (x_1, x_2).$$

Consequences

1. If f is **increasing** on $[a, b]$, then $f(x_2) - f(x_1) > 0$, which implies $f'(x) > 0$ for all $x \in (a, b)$.
2. If f is **decreasing** on $[a, b]$, then $f(x_2) - f(x_1) < 0$, which implies $f'(x) < 0$ for all $x \in (a, b)$.
3. If f is **non-decreasing** on $[a, b]$, then $f'(x) \geq 0$ for all $x \in (a, b)$.
4. If f is **non-increasing** on $[a, b]$, then $f'(x) \leq 0$ for all $x \in (a, b)$.

Example: Monotonicity and the MVT

Let $f(x) = x^3$. Check monotonicity on $[-1, 1]$.

1. Compute $f'(x)$:

$$f'(x) = 3x^2$$

2. Analyze $f'(x)$ on $(-1, 1)$:
   - Since $f'(x) \geq 0$ for all x, $f(x)$ is non-decreasing on $[-1, 1]$.
   - Moreover, $f'(x) > 0$ for all $x \neq 0$, so $f(x)$ is strictly increasing on $[-1, 1]$.
3. Apply the **MVT**:
   For $x_1 = -1$ and $x_2 = 1$, the **MVT** guarantees a point $c \in (-1, 1)$ such that:

$$f'(c) = \frac{f(1) - f(-1)}{1 - (-1)} = \frac{1 - (-1)}{2} = 1$$

Thus, $f'(c) = 1$ at some $c \in (-1, 1)$.

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11. Solutions to Calculus Problems: Preparation for MT 1

📌 **Important Note:**

"I couldn't attend this lecture where students had an open Q&A session with the professor. To ensure I stay on track, I've included the solutions to Self-Study Problems: MT 1 here as a reference."

Problem 1: Evaluate the following limits if they exist. If not, indicate whether the limit does not exist or is $\pm\infty$. Do not use L'Hopital's rule.

(i) $\lim_{x\to 0} \tan\left(\frac{\sin(x)}{x}\right)$

1. **Simplify the argument of** $\tan$ **:**

   As $x \to 0$, the standard limit property states:

   $$\frac{\sin(x)}{x} \to 1$$

   Thus, the argument of $\tan$ approaches:

   $$\tan\left(\frac{\sin(x)}{x}\right) \to \tan(1)$$

2. **Evaluate the limit:**

Since $\tan(1)$ is a finite value, the limit exists and is:

$$\lim_{x \to 0} \tan\left(\frac{\sin(x)}{x}\right) = \tan(1)$$

$(ii)\ \lim_{x \to 3} \frac{|5-2x|-|x-2|}{|x-5|-|3x-7|}$

To solve this, we need to analyze the one-sided limits, as the absolute values could behave differently from either side of $x = 3$.

Step 1: Calculate the Left-Hand Limit $(x \to 3^-)$

For $x < 3$, the absolute values simplify as follows:

- $|5 - 2x| = 5 - 2x$
- $|x - 2| = x - 2$
- $|x - 5| = 5 - x$
- $|3x - 7| = 7 - 3x$

Substituting these expressions into the limit, we get:

$$\lim_{x \to 3^-} \frac{2x - 5 - (x - 2)}{5 - x - (7 - 3x)} = \lim_{x \to 3^-} \frac{x - 3}{-4x + 12}$$

Simplify the expression:

$$= \lim_{x \to 3^-} \frac{x - 3}{-4(x - 3)}$$

Cancel $x - 3$ from the numerator and denominator:

$$= \lim_{x \to 3^-} \frac{1}{-4} = -\frac{1}{4}$$

Step 2: Calculate the Right-Hand Limit $(x \to 3^+)$

For x > 3 , the absolute values simplify as follows:

- $|5 - 2x| = 2x - 5$
- $|x - 2| = x - 2$
- $|x - 5| = x - 5$
- $|3x - 7| = 3x - 7$

Substituting these expressions into the limit, we get:

$$\lim_{x \to 3^+} \frac{2x - 5 - (x - 2)}{5 - x - (3x - 7)} = \lim_{x \to 3^+} \frac{x - 3}{-4x + 12}$$

Simplify the expression:

$$= \lim_{x \to 3^+} \frac{x - 3}{-4(x - 3)}$$

Cancel $x - 3$ from the numerator and denominator:

$$= \lim_{x \to 3^+} \frac{1}{-4} = -\frac{1}{4}$$

Conclusion

Since both the left-hand and right-hand limits are equal:

$$\lim_{x \to 3} \frac{|5 - 2x| - |x - 2|}{|x - 5| - |3x - 7|} = -\frac{1}{4}$$

$(iii) \lim_{x \to \pi} \frac{|x - \pi|}{x^2 - \pi x}$

1. **Simplify the denominator:**

   Rewrite the denominator as:

$$x^2 - \pi x = x(x - \pi)$$

2. **Consider cases for $x > \pi$ and $x < \pi$:**

   - For $x > \pi$, $|x - \pi| = x - \pi$. The expression becomes:

$$\frac{x - \pi}{x(x - \pi)} = \frac{1}{x}$$

     As $x \to \pi^+$, $\frac{1}{x} \to \frac{1}{\pi}$.

   - For $x < \pi$, $|x - \pi| = \pi - x$. The expression becomes:

$$\frac{\pi - x}{x(x - \pi)} = -\frac{1}{x}$$

   As $x \to \pi^-$, $-\frac{1}{x} \to -\frac{1}{\pi}$.

3. **Conclusion:**

   The left-hand and right-hand limits differ. Therefore, the limit **does not exist**.

$(iv) \lim_{x\to-\infty} \frac{3x^3}{\sqrt{x^4-2}}$

1. **Simplify the square root:**

   Factor x^4 inside the square root:

$$\sqrt{x^4-2} = |x^2|\sqrt{1-\frac{2}{x^4}} = x^2\sqrt{1-\frac{2}{x^4}}$$

   since $x \to -\infty$.

2. **Simplify the expression:**

   Substitute the simplified square root into the fraction:

$$\frac{3x^3}{\sqrt{x^4-2}} = \frac{3x^3}{x^2\sqrt{1-\frac{2}{x^4}}} = \frac{3x}{\sqrt{1-\frac{2}{x^4}}}$$

3. **Evaluate the limit:**

   As $x \to -\infty$, the denominator approaches $\sqrt{1} = 1$, so:

$$\frac{3x}{\sqrt{1-\frac{2}{x^4}}} \to 3x$$

   Thus, the limit is:

$$\lim_{x\to-\infty} \frac{3x^3}{\sqrt{x^4-2}} = -\infty$$

$(v) \lim_{x\to0} x^{2/3} \sin\left(\frac{1}{x}\right)$

1. **Bound the oscillatory term:**

   Since $-1 \le \sin\left(\frac{1}{x}\right) \le 1$, multiplying by $x^{2/3}$ gives:

$$-x^{2/3} \le x^{2/3}\sin\left(\frac{1}{x}\right) \le x^{2/3}$$

2. **Apply the Squeeze Theorem:**

   As $x \to 0$, $x^{2/3} \to 0$ and $-x^{2/3} \to 0$. By the Squeeze Theorem:

$$\lim_{x \to 0} x^{2/3} \sin\left(\frac{1}{x}\right) = 0$$

$(vi)\ \lim_{h \to 0} \frac{(h - \frac{1}{3})^2 - \frac{1}{9}}{h}$

1. **Simplify the numerator:**

   Expand $(h - \frac{1}{3})^2 - \frac{1}{9}$:

$$(h - \frac{1}{3})^2 - \frac{1}{9} = h^2 - \frac{2h}{3} + \frac{1}{9} - \frac{1}{9} = h^2 - \frac{2h}{3}$$

2. **Simplify the fraction:**

   Divide the numerator by h :

$$\frac{h^2 - \frac{2h}{3}}{h} = h - \frac{2}{3}$$

3. **Evaluate the limit:**

   As $h \to 0, h - \frac{2}{3} \to -\frac{2}{3}$. Thus:

$$\lim_{h \to 0} \frac{(h - \frac{1}{3})^2 - \frac{1}{9}}{h} = -\frac{2}{3}$$

Problem 2: Find all values of a that make $f(x)$ continuous everywhere.

The function $f(x)$ is defined piecewise as follows:

$$f(x) = \begin{cases} 2x^2 - 3x + 5 & \text{if } x \leq 0 \\ 7\sin^2(3x) + a^2 & \text{if } x > 0 \end{cases}$$

To ensure $f(x)$ is continuous at $x = 0$, we need the limit from the left and right at $x = 0$ to be equal to $f(0)$.

Step 1: Evaluate the Left-Hand Limit $(x \to 0^-)$

For $x \leq 0$, $f(x) = 2x^2 - 3x + 5$. Thus:

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (2x^2 - 3x + 5) = 5$$

Step 2: Evaluate the Right-Hand Limit $(x \to 0^+)$

For $x > 0$, $f(x) = 7\sin^2(3x) + a^2$. As $x \to 0^+$, $\sin^2(3x) \to 0$, so:

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (7 \sin^2(3x) + a^2) = a^2$$

Step 3: Set Limits Equal for Continuity

For $f(x)$ to be continuous at $x = 0$, we need:

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$$

Thus:

$$5 = a^2$$

Step 4: Solve for a

Solving $a^2 = 5$, we get:

$$a = \pm\sqrt{5}$$

Conclusion

The values of a that make $f(x)$ continuous everywhere are:

$$a = \pm\sqrt{5}$$

Problem 3: **Intermediate Value Theorem (IVT)**

(i) Intermediate Value Theorem (IVT) Statement

Let $f : [a, b] \to \mathbb{R}$ be a continuous function on the closed interval $[a, b]$. Suppose that y_0 is any value between $f(a)$ and $f(b)$. In other words, if $y_0 \in (f(a), f(b))$ or $y_0 \in (f(b), f(a))$, then there exists at least one point $x_0 \in [a, b]$ such that $y_0 = f(x_0)$.

This property allows us to conclude that any value between $f(a)$ and $f(b)$ must be attained by the function f at some point within the interval $[a, b]$.

(ii) Application of the IVT

Prove that the equation $\arctan(x) = \frac{\pi}{2} - x$ has at least 1 solution.

Define the function:

$$f(x) := \arctan(x) - \frac{\pi}{2} + x$$

Note that both $\arctan(x)$ and $-\frac{\pi}{2} + x$ are continuous functions over $(-\infty, \infty)$. Consequently, $f(x)$ is continuous over $(-\infty, \infty)$, and particularly on the closed interval $[0, 1]$.

1. **Evaluate $f(0)$:**

$$f(0) = \arctan(0) - \frac{\pi}{2} + 0 = -\frac{\pi}{2}$$

2. **Evaluate $f(1)$:**

$$f(1) = \arctan(1) - \frac{\pi}{2} + 1 = \frac{\pi}{4} - \frac{\pi}{2} + 1 = -\frac{\pi}{4} + 1$$

3. **Check if 0 lies between $f(0)$ and $f(1)$:**

   Since $f(0) = -\frac{\pi}{2} < 0 < -\frac{\pi}{4} + 1 = f(1)$, we observe that 0 lies within the interval $(f(0), f(1)) = (-\frac{\pi}{2}, 1 - \frac{\pi}{4})$.

4. **Conclude by IVT:**

   By the Intermediate Value Theorem, since f is continuous on $[0, 1]$ and 0 lies between $f(0)$ and $f(1)$, there must exist a point $x_0 \in (0, 1)$ such that $f(x_0) = 0$.

   Therefore, we have:

$$f(x_0) = 0 \Rightarrow \arctan(x_0) = \frac{\pi}{2} - x_0$$

Thus, there exists a solution x_0 in $(0, 1)$ that satisfies the equation.

Problem 4: Differentiability and its Definition

(i) Definition of Differentiability at a Point x_0 :

A function $f(x)$ is differentiable at $x = x_0$ if the following limit exists:

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

Alternatively, this can be written by setting $x - x_0 = h$ as:

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

This means that the derivative $f'(x_0)$ represents the slope of the tangent line to the graph of $f(x)$ at $x = x_0$, provided the limit exists.

(ii) General Differentiability of $f(x)$:

A function $f(x)$ is said to be differentiable at a general point x if:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists. This definition implicitly assumes that x is within the domain of $f(x)$ and that f is smooth enough to compute the above limit.

For example:

- Polynomial functions like x^2 or trigonometric functions like $\sin(x)$ are differentiable everywhere within their domains.
- This general definition applies at every point x, but when we specify a particular point $x = a$, it reduces to the case in **part (i)**.

(iii) Differentiability at $x = \frac{1}{2}$:

To check differentiability at $x = \frac{1}{2}$, we use the limit definition:

$$f'\left(\frac{1}{2}\right) = \lim_{h \to 0} \frac{f\left(\frac{1}{2} + h\right) - f\left(\frac{1}{2}\right)}{h}$$

Alternatively, it can also be expressed as:

$$f'\left(\frac{1}{2}\right) = \lim_{x \to \frac{1}{2}} \frac{f(x) - f\left(\frac{1}{2}\right)}{x - \frac{1}{2}}$$

This ensures that the derivative at $x = \frac{1}{2}$ exists if and only if the above limit converges to a finite value.

Problem 5: Differentiability Analysis

Part (a): Compute $f'(x)$ for $f(x) = x^2$ at $x = -1$

1. **Use the definition of the derivative:**

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

2. **Substitute $f(x) = x^2$ and $x_0 = -1$:**

$$f'(-1) = \lim_{h \to 0} \frac{(-1+h)^2 - (-1)^2}{h}$$

3. **Simplify the numerator:**

$$(-1+h)^2 - (-1)^2 = (1 - 2h + h^2) - 1 = -2h + h^2$$

4. **Substitute back into the limit:**

$$f'(-1) = \lim_{h \to 0} \frac{-2h + h^2}{h}$$

5. **Simplify the fraction:**
   Factor h from the numerator:

$$f'(-1) = \lim_{h \to 0} \frac{h(-2 + h)}{h}$$

   Cancel h (valid for $h \neq 0$):

$$f'(-1) = \lim_{h \to 0} (-2 + h)$$

6. **Evaluate the limit:**

$$f'(-1) = -2$$

Part (b): Compute $f'(0) for f(x) = \frac{2-x}{2+x}$

1. **Set up the derivative using the definition**:

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x}$$

2. **Evaluate** $f(0)$ **:**

$$f(0) = \frac{2-0}{2+0} = 1$$

3. **Substitute into the definition**:

$$f'(0) = \lim_{x \to 0} \frac{\frac{2-x}{2+x} - 1}{x}$$

4. **Simplify:**

$$f'(0) = \lim_{x \to 0} \frac{\frac{-2x}{2+x}}{x} = \lim_{x \to 0} \frac{-2}{2+x}$$

5. **Evaluate the limit**:

$$f'(0) = \frac{-2}{2} = -1$$

Part (c): Prove the Derivative of $f(x) = \cos(x)$

1. **Use the definition of the derivative**:

$$f'(x_0) = \lim_{x \to x_0} \frac{\cos(x) - \cos(x_0)}{x - x_0}$$

2. **Simplify using trigonometric identity**:
   Use $\cos(A) - \cos(B) = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$:

$$f'(x_0) = \lim_{x \to x_0} \frac{-2\sin\left(\frac{x+x_0}{2}\right)\sin\left(\frac{x-x_0}{2}\right)}{x - x_0}$$

3. **Substitute** $h = x - x_0$:

   Rewrite the limit:

$$f'(x_0) = \lim_{h \to 0} -\sin\left(x_0\right) \cdot \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}$$

4. **Simplify**:

   Using $\lim_{h \to 0} \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} = 1$:

$$f'(x_0) = -\sin(x_0)$$

Problem 6: Differentiability of $f(x) = |x^2 - 9|$

Part (a): For what values of x is $f(x)$ differentiable? Find a formula for $f'(x)$.

1. **Analyze the definition of $f(x) = |x^2 - 9|$:**

   Rewrite $f(x)$ using its piecewise definition:

$$f(x) = \begin{cases} x^2 - 9 & \text{if } x^2 \geq 9, \text{ i.e., } x \geq 3 \text{ or } x \leq -3, \\ 9 - x^2 & \text{if } -3 < x < 3. \end{cases}$$

2. **Differentiate each piece:**

   - For $x^2 \geq 9$:

$$f'(x) = \frac{d}{dx}(x^2 - 9) = 2x$$

- For $-3 < x < 3$:

$$f'(x) = \frac{d}{dx}(9 - x^2) = -2x$$

3. **Combine into a formula:**

$$f'(x) = \begin{cases} 2x & \text{if } x > 3 \text{ or } x < -3, \\ -2x & \text{if } -3 < x < 3. \end{cases}$$

4. **Check differentiability at $x = 3$ and $x = -3$:**

   - The left-hand derivative at $x = 3$:

$$f'(3) = \lim h \to 0^- \frac{f(3+h) - f(3)}{h} = \lim_{h \to 0^-} \frac{(3+h)^2 - 9}{h} = 6$$

   - The right-hand derivative at $x = 3$:

$$f' + (3) = \lim h \to 0^+ \frac{f(3+h) - f(3)}{h} = \lim_{h \to 0^+} \frac{(3+h)^2 - 9}{h} = 6$$

   Since $f'_-(3) \neq f'_+(3)$, $f(x)$ is **not differentiable at** $x = 3$.

   Similarly, at $x = -3$:

$$f'_-(3) \neq f'_+(3)$$

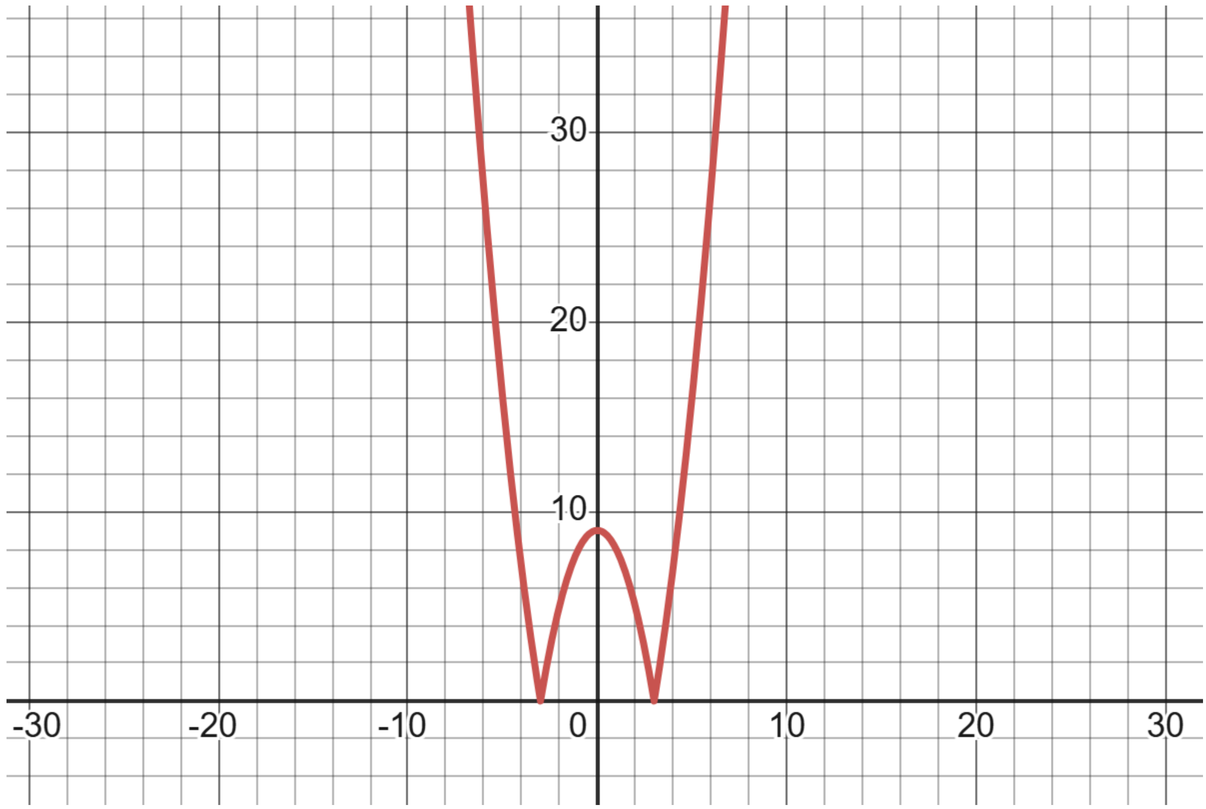
   so $f(x)$ is also **not differentiable at** $x = -3$.

5. **Conclusion:**

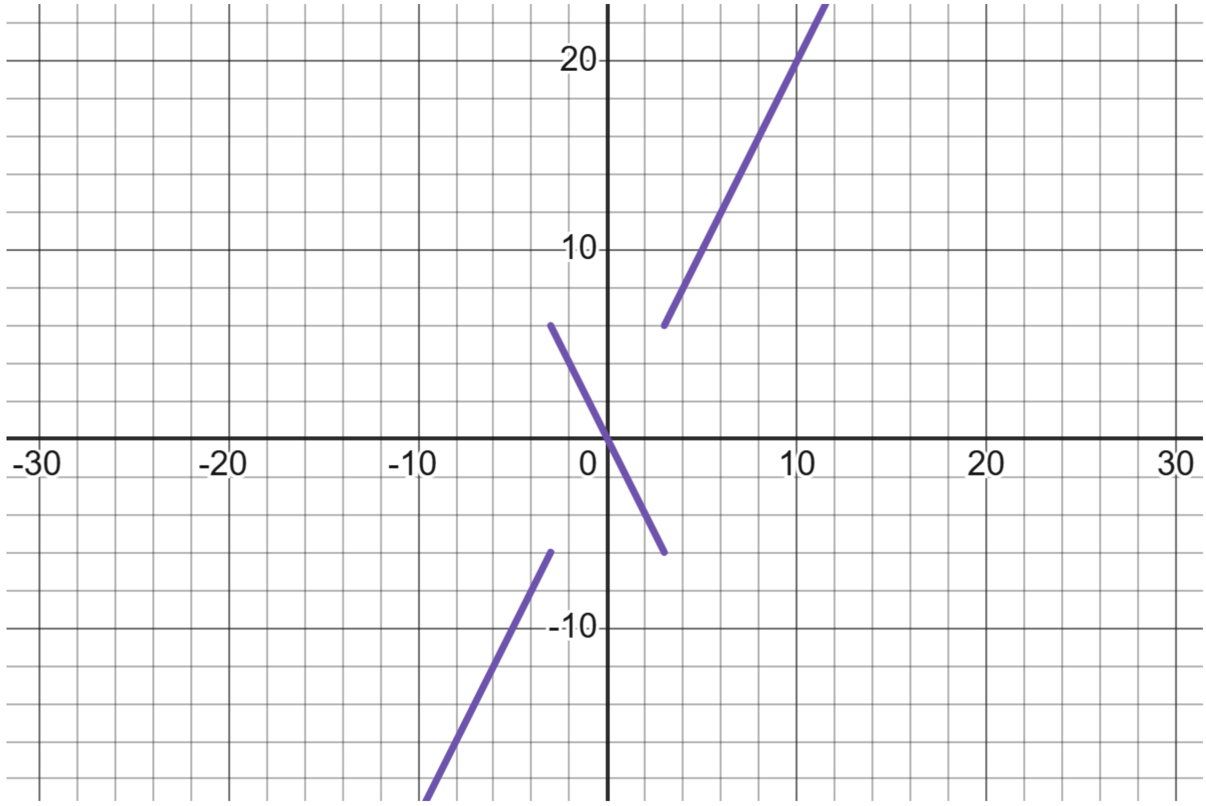
   $f(x) = |x^2 - 9|$ is differentiable for $x \in \mathbb{R} \setminus \{-3, 3\}$.

Part (b): Sketch the Graphs of $f(x)$ and $f'(x)$

1. **Graph of $f(x)$:**



2. **Graph of $f'(x)$:**



Problem 7: Analyze the Differentiability of $g(x) = |x-1| + |x+2|$

(a) For what values of x is $g(x)$ differentiable?

1. **Piecewise Definition of $g(x)$:**

   The absolute value terms introduce critical points at $x = 1$ and $x = -2$. Let us rewrite $g(x)$ piecewise:

   - For $x > 1$, $|x-1| = x-1$ and $|x+2| = x+2$, so:

   $$g(x) = (x-1) + (x+2) = 2x+1$$

   - For $-2 \leq x \leq 1$, $|x-1| = 1-x$ and $|x+2| = x+2$, so:

   $$g(x) = (1-x) + (x+2) = 3$$

   - For $x < -2$, $|x-1| = 1-x$ and $|x+2| = -x-2$, so:

   $$g(x) = (1-x) + (-x-2) = -2x-1$$

   The piecewise definition is:

   $$g(x) = \begin{cases} 2x+1 & \text{if } x > 1, \\ 3 & \text{if } -2 \leq x \leq 1, \\ -2x-1 & \text{if } x < -2. \end{cases}$$

2. **Derivative in Each Interval:**

   - For $x > 1$:

   $$g'(x) = \frac{d}{dx}(2x+1) = 2$$

   - For $-2 \leq x \leq 1$:

   $$g'(x) = \frac{d}{dx}(3) = 0$$

   - For $x < -2$:

   $$g'(x) = \frac{d}{dx}(-2x-1) = -2$$

The derivative is:

$$g'(x) = \begin{cases} 2 & \text{if } x > 1, \\ 0 & \text{if } -2 < x < 1, \\ -2 & \text{if } x < -2. \end{cases}$$

3. **Check Differentiability at** $x = 1$ **:**

   - Left-hand derivative:

$$g'_-(1) = \lim h \to 0^- \frac{g(1+h) - g(1)}{h} = \lim_{h \to 0^-} \frac{3-3}{h} = 0$$

   - Right-hand derivative:

$$g'_+(1) = \lim h \to 0^+ \frac{g(1+h) - g(1)}{h} = \lim_{h \to 0^+} \frac{(2(1+h)+1) - 3}{h} = 2$$

   Since $g'(1) \neq g' + (1)$, $g(x)$ is **not differentiable at** $x = 1$.

4. **Check Differentiability at** $x = -2$ **:**

   - Left-hand derivative:

$$g'_-(2) = \lim h \to 0^- \frac{g(-2+h) - g(-2)}{h} = \lim_{h \to 0^-} \frac{(-2(-2+h)-1) - 3}{h} = -2$$

   - Right-hand derivative:

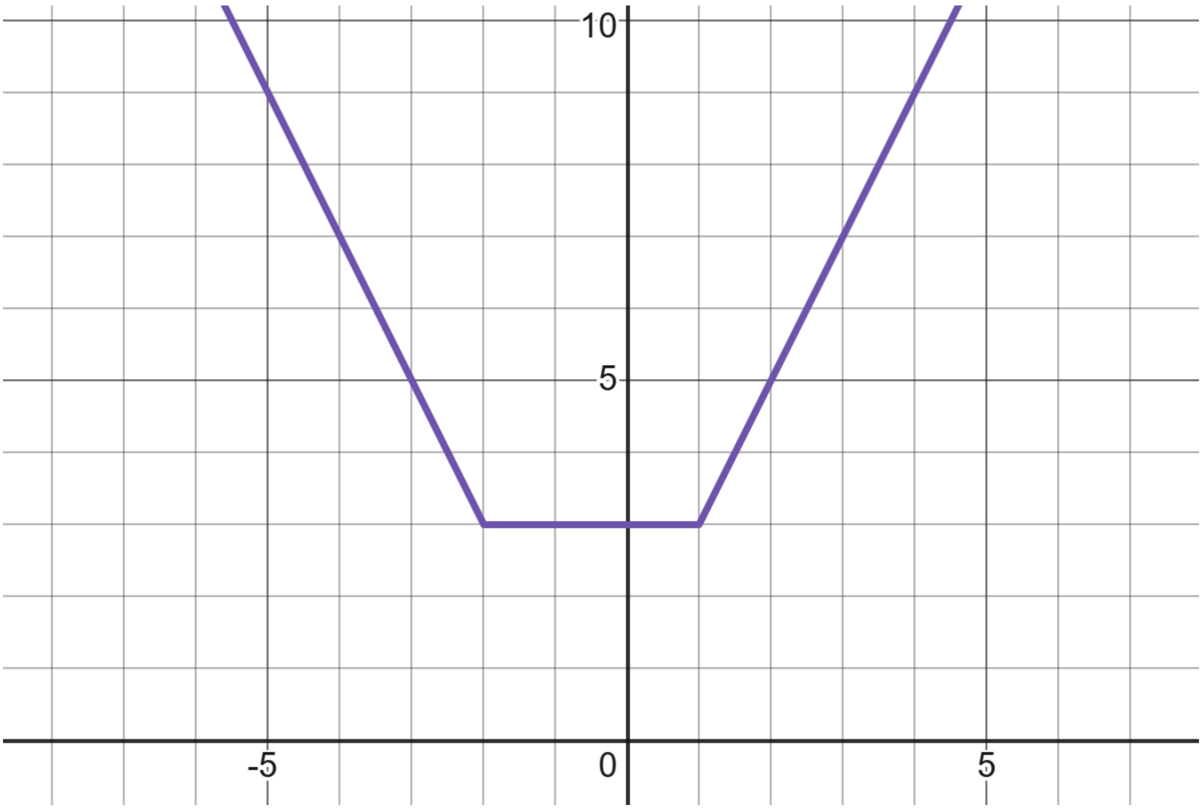
$$g'_+(2) = \lim h \to 0^+ \frac{g(-2+h) - g(-2)}{h} = \lim_{h \to 0^+} \frac{3-3}{h} = 0$$

   Since $g'(2) \neq g' + (2)$, $g(x)$ is **not differentiable at** $x = -2$.

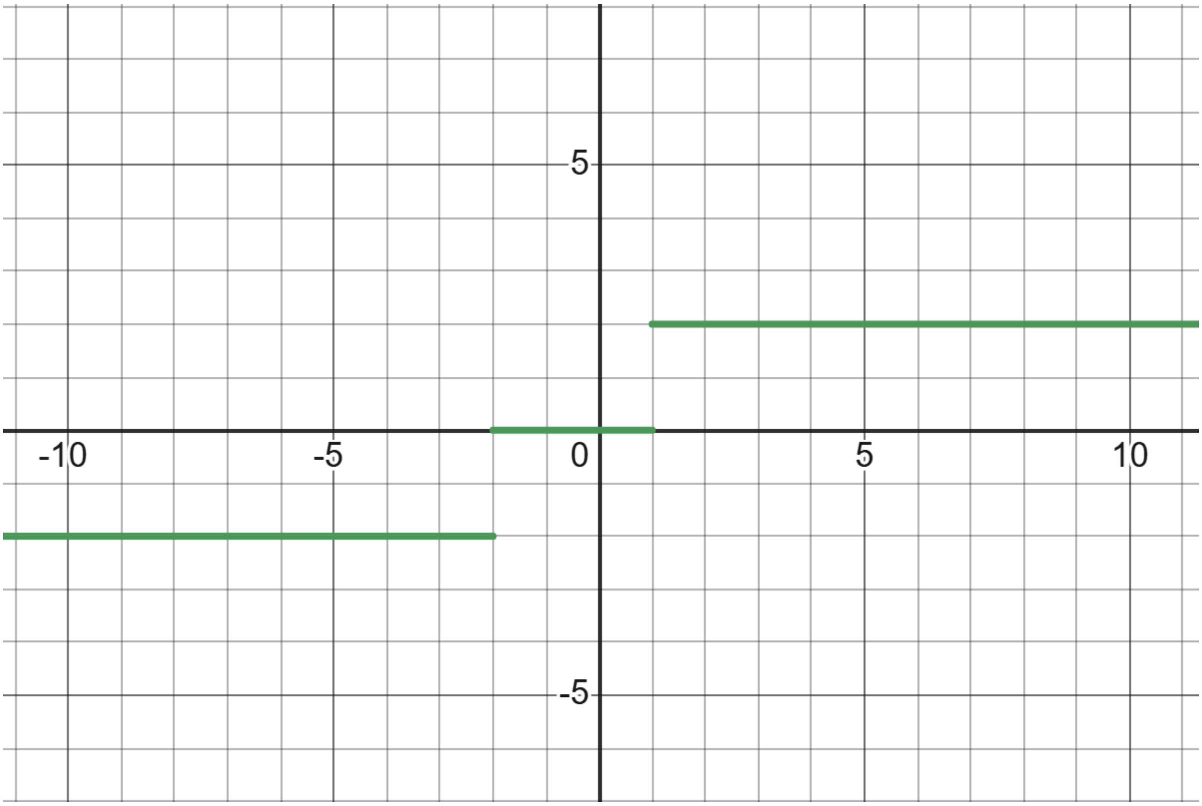
Final Answer for (a): $g(x)$ **is differentiable for** $x \in \mathbb{R} \setminus \{-2, 1\}$**.**

Part (b): Sketch $g(x)$ **and** $g'(x)$

1. **Graph of** $g(x)$ **:**



2. **Graph of** $g'(x)$ **:**



Problem 8: Compute the Following Derivatives

(i) Find $\frac{ds}{dt}$ for $s(t) = \sqrt[5]{\cos(\sqrt{t})}$

1. Rewrite $s(t)$:

$$s(t) = \cos(\sqrt{t})^{1/5}$$

2. Apply the chain rule:

$$\frac{ds}{dt} = \frac{1}{5}\cos(\sqrt{t})^{-4/5} \cdot \frac{d}{dt}[\cos(\sqrt{t})]$$

3. Differentiate \cos(\sqrt{t}) :

$$\frac{d}{dt}[\cos(\sqrt{t})] = -\sin(\sqrt{t}) \cdot \frac{1}{2\sqrt{t}}$$

4. Combine:

$$\frac{ds}{dt} = \frac{1}{5}\cos(\sqrt{t})^{-4/5} \cdot \left(-\sin(\sqrt{t}) \cdot \frac{1}{2\sqrt{t}}\right)$$

5. Simplify:

$$\frac{ds}{dt} = -\frac{\sin(\sqrt{t})}{10\sqrt{t}\cos(\sqrt{t})^{4/5}}$$

(ii) Find $\frac{df}{dt}$ for $f(t) = \sin(\cos(4t))$

1. Apply the chain rule:

$$\frac{df}{dt} = \cos(\cos(4t)) \cdot \frac{d}{dt}[\cos(4t)]$$

2. Differentiate $\cos(4t)$:

$$\frac{d}{dt}[\cos(4t)] = -\sin(4t) \cdot 4$$

3. Combine:

$$\frac{df}{dt} = \cos(\cos(4t)) \cdot (-4\sin(4t))$$

4. Simplify:

$$\frac{df}{dt} = -4\cos(\cos(4t))\sin(4t)$$

(iii) Compute $\frac{d^2}{dx^2}\left[(x^2+3)\sqrt{x}\right]$

1. Rewrite:

$$(x^2+3)\sqrt{x} = (x^2+3)x^{1/2}$$

2. First derivative using the product rule:

$$\frac{d}{dx}[(x^2+3)x^{1/2}] = \frac{d}{dx}[x^2+3]\cdot x^{1/2} + (x^2+3)\cdot\frac{d}{dx}[x^{1/2}]$$

   - $\frac{d}{dx}[x^2+3] = 2x.$
   - $\frac{d}{dx}[x^{1/2}] = \frac{1}{2\sqrt{x}}.$

   Substitute:

$$\frac{d}{dx}[(x^2+3)x^{1/2}] = 2x\cdot x^{1/2} + (x^2+3)\cdot\frac{1}{2\sqrt{x}}$$

3. Simplify:

$$\frac{d}{dx}[(x^2+3)x^{1/2}] = 2x^{3/2} + \frac{x^2+3}{2\sqrt{x}}$$

4. Second derivative:
   Part (a): Differentiate $2x^{3/2} + \frac{x^2+3}{2\sqrt{x}}$ term by term:

   - For $2x^{3/2}$, the derivative is:

$$\frac{d}{dx}[2x^{3/2}] = 3x^{1/2}$$

   - For $\frac{x^2+3}{2\sqrt{x}}$, simplify to $\frac{x^2}{2\sqrt{x}} + \frac{3}{2\sqrt{x}} = \frac{x^{3/2}}{2} + \frac{3}{2}x^{-1/2}$, then differentiate:

$$\frac{d}{dx}\left(\frac{x^{3/2}}{2}\right) = \frac{3}{4}x^{1/2}, \quad \frac{d}{dx}\left(\frac{3}{2}x^{-1/2}\right) = -\frac{3}{4}x^{-3/2}$$

5. Combine:

$$\frac{d^2}{dx^2}[(x^2+3)\sqrt{x}] = 3x^{1/2} + \frac{3}{4}x^{1/2} - \frac{3}{4}x^{-3/2}$$

6. Simplify:

$$\frac{d^2}{dx^2}[(x^2+3)\sqrt{x}] = \frac{15}{4}x^{1/2} - \frac{3}{4}x^{-3/2}$$

Part (b): Compute $g''(x) = \frac{d^2}{dx^2}\left[\frac{\sin(x)}{x}\right]$.

1. **Set up the first derivative** $g'(x)$ **:**
   Using the quotient rule:

$$g'(x) = \frac{\frac{d}{dx}[\sin(x)] \cdot x - \sin(x) \cdot \frac{d}{dx}[x]}{x^2}$$

- $\frac{d}{dx}[\sin(x)] = \cos(x),$
- $\frac{d}{dx} = 1.$

Substituting:

$$g'(x) = \frac{\cos(x) \cdot x - \sin(x) \cdot 1}{x^2}$$

Simplify:

$$g'(x) = \frac{x\cos(x) - \sin(x)}{x^2}$$

2. **Compute the second derivative** $g''(x)$ **:**
   Differentiate $g'(x)$ using the quotient rule again:

$$g''(x) = \frac{\frac{d}{dx}[x\cos(x) - \sin(x)] \cdot x^2 - (x\cos(x) - \sin(x)) \cdot \frac{d}{dx}[x^2]}{(x^2)^2}$$

3. **Differentiate** $x\cos(x) - \sin(x)$ **:**
   Using the product rule for $x\cos(x)$:

$$\frac{d}{dx}[x\cos(x)] = \cos(x) - x\sin(x)$$

and for $-\sin(x)$:

$$\frac{d}{dx}[-\sin(x)] = -\cos(x)$$

So:

$$\frac{d}{dx}[x\cos(x) - \sin(x)] = \cos(x) - x\sin(x) - \cos(x) = -x\sin(x)$$

4. **Simplify the second derivative:**
   Substituting back:

$$g''(x) = \frac{(-x\sin(x)) \cdot x^2 - (x\cos(x) - \sin(x)) \cdot 2x}{x^4}$$

   Simplify the numerator:

$$g''(x) = \frac{-x^3\sin(x) - 2x(x\cos(x) - \sin(x))}{x^4}$$

   Expand:

$$g''(x) = \frac{-x^3\sin(x) - 2x^2\cos(x) + 2x\sin(x)}{x^4}$$

5. **Factorize:**

$$g''(x) = \frac{-x^2\sin(x) - 2x\cos(x) + 2\sin(x)}{x^3}$$

(iv) Compute $\frac{d}{dz}((z+1)^{2024}\tan(3z))$

1. Apply the product rule:

$$\frac{d}{dz}((z+1)^{2024}\tan(3z)) = \frac{d}{dz}[(z+1)^{2024}] \cdot \tan(3z) + (z+1)^{2024} \cdot \frac{d}{dz}[\tan(3z)]$$

2. Differentiate each term:
   - $\frac{d}{dz}[(z+1)^{2024}] = 2024(z+1)^{2023}$.
   - $\frac{d}{dz}[\tan(3z)] = 3\sec^2(3z)$.

3. Combine:

$$\frac{d}{dz}((z+1)^{2024}\tan(3z)) = 2024(z+1)^{2023}\tan(3z) + 3(z+1)^{2024}\sec^2(3z).$$

(v) Find $\frac{dy}{dx}\Big|_{t=\frac{\pi}{4}}$, where $x(t) = t^2 + 2$, $y(t) = \tan(t) - 3$

1. Use the chain rule:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

2. Differentiate $x(t)$ and $y(t)$:
   - $\frac{dx}{dt} = 2t.$
   - $\frac{dy}{dt} = \sec^2(t).$
3. Substitute:

$$\frac{dy}{dx} = \frac{\sec^2(t)}{2t}$$

4. Evaluate at $t = \frac{\pi}{4}$:
   - $\sec^2\left(\frac{\pi}{4}\right) = 2.$
   - $2t = \frac{\pi}{2}.$

   Substitute:

$$\frac{dy}{dx} = \frac{2}{\frac{\pi}{2}} = \frac{4}{\pi}$$

Problem 9: Theorems and Their Applications

Part (a): Rolle's Theorem

Statement:

Let f be a continuous function on the closed interval $[a, b]$, differentiable on the open interval (a, b), and satisfying $f(a) = f(b)$. Then, there exists at least one point $c \in (a, b)$ such that:

$$f'(c) = 0$$

Part (b): The Mean Value Theorem (MVT)

Statement:

Let f be a continuous function on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Then, there exists at least one point $c \in (a, b)$ such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Part (c): Show that the equation $2x - 1 = \sin(x)$ has exactly one solution in the open interval $(0, \pi)$.

1. **Existence of a Solution**

   Define $f(x) = 2x - 1 - \sin(x)$.

   - $f(x)$ is continuous on $[0, \pi]$ because it is the difference of continuous functions.
   - Evaluate at endpoints:

   $$f(0) = -1 \quad \text{and} \quad f(\pi) = 2\pi - 1 - \sin(\pi) = 2\pi - 1 > 0$$

   - By the **Intermediate Value Theorem (IVT)**, there exists $x_1 \in (0, \pi)$ such that $f(x_1) = 0$.

2. **Uniqueness of the Solution**

   Assume there are two distinct solutions, x_1 and x_2, such that $f(x_1) = f(x_2) = 0$. By Rolle's Theorem:

   - Since $f(x_1) = f(x_2)$, there exists $c \in (x_1, x_2)$ such that $f'(c) = 0$.
   - Compute $f'(x)$:

   $$f'(x) = 2 - \cos(x)$$

   - Since $-1 \leq \cos(x) \leq 1$ for $x \in (0, \pi)$, we have $f'(x) = 2 - \cos(x) > 1$.
   - This contradicts $f'(c) = 0$, so there cannot be two distinct solutions.

Conclusion: The equation has exactly one solution in $(0, \pi)$.

Part (d): Prove $0 < f(5) < 3$ given f is differentiable and $f(2) = -3$, $1 < f'(x) < 2$ for $x \in (2, 5)$.

1. **Apply MVT:**

   Since f is differentiable on $(2, 5)$ and continuous on $[2, 5]$, there exists $c \in (2, 5)$ such that:

   $$f'(c) = \frac{f(5) - f(2)}{5 - 2} = \frac{f(5) + 3}{3}$$

   - Given $1 < f'(c) < 2$, we have:

$$1 < \frac{f(5)+3}{3} < 2$$

2. **Simplify the inequalities:**

   Multiply through by 3:

$$3 < f(5) + 3 < 6$$

   Subtract 3:

$$0 < f(5) < 3$$

Part (e): Show that $\tan(x) > x$ for $0 < x < \frac{\pi}{2}$.

1. **Define a function:**

   Let $f(x) = \tan(x) - x$. Then, $f(x)$ is differentiable on $(0, \frac{\pi}{2})$.

2. **Compute the derivative:**

$$f'(x) = \sec^2(x) - 1 = \tan^2(x)$$

   - For $0 < x < \frac{\pi}{2}$, $\tan^2(x) > 0$, so $f'(x) > 0$.

3. **Conclusion:**

   Since $f'(x) > 0$, $f(x) = \tan(x) - x$ is increasing on $(0, \frac{\pi}{2})$.

   - At $x = 0$, $f(0) = 0$. Therefore, $f(x) > 0$ for $0 < x < \frac{\pi}{2}$.
   - Hence, $\tan(x) > x$ for $0 < x < \frac{\pi}{2}$.

Part (f): Discuss whether $h(x) = |x|$ contradicts MVT.

1. **Conditions for MVT:**

   $h(x) = |x|$ is continuous on $[-1, 1]$ but not differentiable at $x = 0$.

2. **Conclusion:**

   MVT requires differentiability on the open interval $(-1, 1)$. Since $h(x)$ is not differentiable at $x = 0$, **MVT** does not apply, and there is no contradiction.

Problem 10 Solutions

(a) Tangent Line to $y = \sqrt[4]{x}$ at $(2, \sqrt[4]{2})$

Step 1: Find the slope of the tangent line

The given function is $y = \sqrt[4]{x}$, which can be written as:

$$y = x^{1/4}$$

Differentiate y with respect to x :

$$y'(x) = \frac{1}{4}x^{-3/4}$$

Substitute $x = 2$ into $y'(x)$:

$$y'(2) = \frac{1}{4}(2)^{-3/4}$$

Simplify $(2)^{-3/4}$ using exponent rules:

$$(2)^{-3/4} = \frac{1}{\sqrt[4]{2^3}} = \frac{1}{\sqrt[4]{8}}$$

Thus:

$$y'(2) = \frac{1}{4 \cdot \sqrt[4]{8}}$$

Step 2: Equation of the tangent line

The equation of the tangent line is:

$$y - y_0 = m(x - x_0)$$

where $m = y'(2) = \frac{1}{4 \cdot \sqrt[4]{8}}$, $x_0 = 2$, and $y_0 = \sqrt[4]{2}$.

Substitute these values:

$$y - \sqrt[4]{2} = \frac{1}{4 \cdot \sqrt[4]{8}}(x - 2)$$

Simplify:

$$y = \sqrt[4]{2} + \frac{1}{4 \cdot \sqrt[4]{8}}(x - 2)$$

Final Answer:

The equation of the tangent line is:

$$y=\sqrt[4]{2}+\frac{x-2}{4\cdot\sqrt[4]{8}}$$

(b) Find a Parabola $y=ax^2+bx$ Such That Its Tangent Line at $(1,1)$ is $y=3x-2$

1. **Conditions**:
   - The point $(1,1)$ lies on the parabola:

     $$a(1)^2+b(1)=1$$

     This simplifies to:

     $$a+b=1 \tag{1}$$

   - The slope of the tangent line at x = 1 is equal to 3:

     $$\frac{d}{dx}\left(ax^2+bx\right)=2ax+b$$

     At $x=1:$

     $$2a(1)+b=3$$

     This simplifies to:

     $$2a+b=3 \tag{2}$$

2. **Solve the System of Equations**:

   From (1) and $(2):$

   Subtract $(1)$from $(2):$

   $$2a+b-(a+b)=3-1$$

   $$a=2$$

   Substitute $a=2$ into $(1):$

   $$2+b=1$$

   $$b=-1$$

3. **Equation of the Parabola**:

The equation is:

$$y = 2x^2 - x$$

12. Applications of Derivatives

1. Extreme Values

Definitions

Let $f : I \to \mathbb{R}$ be a function defined on an interval I :

- **Local Minimum:**

  $(c, f(c))$ is a local minimum of f if:

  $$f(c) \le f(x), \quad \forall x \in (c - \delta, c + \delta) \cap I$$

  for some $\delta > 0.$

- **Local Maximum:**

  $(c, f(c))$ is a local maximum of f if:

  $$f(c) \ge f(x), \quad \forall x \in (c - \delta, c + \delta) \cap I$$

  for some $\delta > 0.$

- **Global (Absolute) Minimum:**

  $(c, f(c))$ is a global minimum of f on I if:

$$f(c) \leq f(x), \quad \forall x \in I$$

- **Global (Absolute) Maximum:**

  $(c, f(c))$ is a global maximum of f on I if:

$$f(c) \geq f(x), \quad \forall x \in I$$

Extreme Value Theorem (EVT)

The **Extreme Value Theorem** states:

If $f(x)$ is **continuous** on a **closed interval** $[a, b]$, then:

1. $f(x)$ attains both an **absolute maximum** and an **absolute minimum** on $[a, b]$.
2. That is, there exist points c and d in $[a, b]$ such that:
   - $f(c) \geq f(x)$ for all $x \in [a, b]$ (**absolute maximum**).
   - $f(d) \leq f(x)$ for all $x \in [a, b]$ (**absolute minimum**).

Key Points

1. The function must be **continuous** on the **closed interval** $[a, b]$.
2. The absolute maximum and minimum values may occur at:
   - **Endpoints** of the interval a or b, or
   - **Critical points** where $f'(x) = 0$ or $f'(x)$ is undefined.

Why the Theorem is True

1. **Continuity on a Closed Interval**:
   - A continuous function on $[a, b]$ is bounded and does not "blow up" to infinity.
2. **Compactness of $[a, b]$** :
   - The interval $[a, b]$ is compact (closed and bounded), ensuring that $f(x)$ has both a greatest and a least value on this interval.

2. Critical Points and Local Extrema

Theorem: Critical Points and Local Extrema

Let $f : I \to \mathbb{R}$ be a function, and $(c, f(c))$ is a **local extremum** of f. Then c is one of the following:

1. A **critical point**: $f'(c) = 0$
2. A **singular point**: $f'(c)$ does not exist
3. An **endpoint** of the interval I.

3. First Derivative Test

The **First Derivative Test** is used to determine whether a critical point is a local maximum, local minimum, or neither.

Theorem: First Derivative Test

Suppose c is a critical point of f and $f'(x)$ changes sign around c :

1. If $f'(x)$ changes from **positive to negative** at c, $f(c)$ is a **local maximum**.
2. If $f'(x)$ changes from **negative to positive** at c, $f(c)$ is a **local minimum**.
3. If $f'(x)$ does not change sign at c, $f(c)$ is **neither a local maximum nor a local minimum**.

Example

Let $f(x) = x^3 - 3x^2$. Find local extrema:

1. Find $f'(x) = 3x^2 - 6x$.
2. Solve $f'(x) = 0 : x = 0, x = 2$.
3. Analyze the sign of $f'(x)$:
   - $f'(x) > 0$ for $x < 0$,
   - $f'(x) < 0$ for $0 < x < 2$,
   - $f'(x) > 0$ for $x > 2$.
4. Conclusion:
   - $x = 0$: Local maximum.

- $x = 2$: Local minimum.

4. Second Derivative Test

The **Second Derivative Test** determines the nature of a critical point based on concavity.

Theorem: Second Derivative Test

Let f be a twice-differentiable function. If c is a critical point $(f'(c) = 0)$, then:

1. If $f''(c) > 0$, $f(c)$ is a **local minimum**.
2. If $f''(c) < 0$, $f(c)$ is a **local maximum**.
3. If $f''(c) = 0$, the test is inconclusive.

Example

Let $f(x) = x^4 - 4x^2$. Find local extrema:

1. Find $f'(x) = 4x^3 - 8x.$
2. Solve $f'(x) = 0 : x = 0, x = -\sqrt{2}, x = \sqrt{2}.$
3. Find $f''(x) = 12x^2 - 8.$
4. Evaluate $f''(x)$ at critical points:
   - $x = 0 : f''(0) = -8$, local maximum.
   - $x = \pm\sqrt{2} : f''(\pm\sqrt{2}) = 16$, local minimums.

Example: Absolute Extrema of a Continuous Function

Let $f(x) = |x^2 - 4|$ on $[-3, 3]$.

1. **Endpoints**:
   - $f(-3) = |9 - 4| = 5,$
   - $f(3) = |9 - 4| = 5.$
2. **Critical Points**:
   Solve $f'(x) = 0 :$
   - $f'(x) = 2x$, so $x = 0.$

Evaluate $f(0) = |0 - 4| = 4$.

3. **Absolute Extrema**:
   - Absolute Minimum: $(0, 4)$,
   - Absolute Maximum: $(-3, 5), (3, 5)$.

Remark: Critical Points and Extrema

- Not every **critical point** or **singular point** corresponds to a local extremum.
- For example, let $f(x) = x^3$. The derivative $f'(x) = 3x^2$ has a critical point at $x = 0$. However, $f(x)$ has no local maximum or minimum at $x = 0$ because $f(x)$ is increasing both before and after $x = 0$.

---

13. Derivative Tests and Concavity

Second Derivative and Its Role in Concavity

Definitions

1. **Concave Up:**
   A function $f(x)$ is said to be **concave up** on an interval (a, b) if the graph of $f(x)$ lies above all its tangent lines within (a, b).

   Mathematically, f is concave up if:

   $$f''(x) > 0 \quad \forall x \in (a, b).$$

2. **Concave Down:**
   A function $f(x)$ is said to be **concave down** on an interval (a, b) if the graph of $f(x)$ lies below all its tangent lines within (a, b).

   Mathematically, f is concave down if:

   $$f''(x) < 0 \quad \forall x \in (a, b).$$

3. **Inflection Point:**
   A point c is an **inflection point** of $f(x)$ if $f(x)$ changes concavity at c.

   In other words, $f(x)$ transitions from concave up to concave down (or vice versa) at c.

   Necessary condition for c to be an inflection point:

$$f''(c) = 0 \quad \text{or} \quad f''(c) \text{ does not exist.}$$

Derivative Tests

First Derivative Test (Recap)

The **First Derivative Test** determines whether a critical point is a local extremum:

1. If $f'(x)$ changes from **positive to negative** at c, $f(c)$ is a **local maximum**.
2. If $f'(x)$ changes from **negative to positive** at c, $f(c)$ is a **local minimum**.
3. If $f'(x)$ does not change sign at c, $f(c)$ is **neither a local maximum nor a local minimum**.

Second Derivative Test

The **Second Derivative Test** is used to determine the nature of a critical point based on concavity:

1. If $f''(c) > 0, f(c)$ is a **local minimum** (concave up at c).
2. If $f''(c) < 0, f(c)$ is a **local maximum** (concave down at c).
3. If $f''(c) = 0$, the test is **inconclusive**, and other methods (like the **First Derivative Test**) must be used.

Concavity Analysis on an Interval

To determine concavity:

1. Find the second derivative, $f''(x)$.
2. Solve $f''(x) = 0$ or $f''(x) \text{ does not exist}$ to identify potential inflection points.
3. Test the sign of $f''(x)$ in each subinterval formed by the critical points of $f''(x)$:

- If $f''(x) > 0$, $f(x)$ is **concave up** on that interval.
- If $f''(x) < 0$, $f(x)$ is **concave down** on that interval.

Example 1: Determine Concavity and Inflection Points

Let $f(x) = x^3 - 3x^2 + 4$.

1. Compute the first derivative:

$$f'(x) = 3x^2 - 6x$$

2. Compute the second derivative:

$$f''(x) = 6x - 6$$

3. Solve $f''(x) = 0$ to find potential inflection points:

$$6x - 6 = 0 \implies x = 1$$

4. Test the sign of $f''(x)$ in intervals $(-\infty, 1)$ and $(1, \infty)$:
   - For $x < 1 : f''(x) = 6x - 6 < 0$, so $f(x)$ is **concave down**.
   - For $x > 1 : f''(x) = 6x - 6 > 0$, so $f(x)$ is **concave up**.
5. Inflection Point:

   Since $f''(x)$ changes sign at $x = 1$, $(1, f(1))$ is an inflection point:

$$f(1) = 1^3 - 3(1)^2 + 4 = 2.$$

   Inflection Point: $(1, 2)$.

Example 2: Application of Second Derivative Test

Let $f(x) = x^4 - 4x^2$.

1. Compute $f'(x)$ and $f''(x)$:

$$f'(x) = 4x^3 - 8x, \quad f''(x) = 12x^2 - 8$$

2. Find critical points by solving $f'(x) = 0$:

$$4x(x^2 - 2) = 0 \implies x = 0, x = \pm\sqrt{2}.$$

3. Use the Second Derivative Test:

   - $f''(0) = -8$: Concave down, $x = 0$ is a local maximum.
   - $f''(\pm\sqrt{2}) = 16$: Concave up, $x = \pm\sqrt{2}$ are local minima.

4. Conclusion:

   - Local maximum: $(0, 0)$.
   - Local minima: $(\sqrt{2}, -4)$, $(-\sqrt{2}, -4)$.

Remark: Inflection Points

Inflection points occur where the concavity of a function changes. These points are not necessarily critical points (i.e., $f'(x) \neq 0$ at an inflection point).

For example, $f(x) = x^3$ has an inflection point at $x = 0$, but $f'(0) = 0$ and $f''(0) = 0$, yet concavity changes from down to up.

---

14. Sketching the Graph of a Function

Procedure for Graph Sketching

To sketch the graph of a function $f(x)$, follow these systematic steps:

Step 1: Determine the Domain and Intercepts

1. **Domain**: Identify all values of x for which $f(x)$ is defined. Exclude any points where the denominator equals zero or the function involves undefined operations (e.g., square root of a negative number).

2. **Intercepts**:

   - **x-intercepts**: Solve $f(x) = 0$.

   - **y-intercept**: Evaluate $f(0)$ if $x = 0$ is in the domain.

Step 2: Asymptotes

1. **Vertical Asymptotes**: Solve for x where the denominator of $f(x)$ equals zero (if $f(x)$ diverges at those points).

2. **Horizontal Asymptotes**: Check the behavior of $f(x)$ as $x \to \pm\infty$. This often involves finding limits:

$$\lim_{x\to\infty} f(x) \quad \text{and} \quad \lim_{x\to-\infty} f(x).$$

Step 3: First Derivative Analysis ($f'(x)$)

1. **Critical Points**:
   - Solve $f'(x) = 0$ to find where the slope is zero (local maxima/minima candidates).
   - Identify points where $f'(x)$ does not exist (singular points).
2. **Intervals of Increase and Decrease**:
   - Determine the sign of $f'(x)$ on each interval separated by critical points. If $f'(x) > 0$, the function is increasing; if $f'(x) < 0$, the function is decreasing.
3. **Classification**:
   - Use the **First Derivative Test** to classify critical points as local maxima, minima, or neither.

Step 4: Second Derivative Analysis ($f''(x)$)

1. **Intervals of Concavity**:
   - Determine the sign of $f''(x)$ on each interval. If $f''(x) > 0$, the graph is concave up; if $f''(x) < 0$, the graph is concave down.
2. **Inflection Points**:
   - Solve $f''(x) = 0$ or find where $f''(x)$ changes sign. These points indicate a change in concavity.

Step 5: Sketch the Graph

1. Combine all the information from Steps 1-4.
2. Plot:
   - Domain restrictions and asymptotes.

- Intervals of increase, decrease, concavity, and inflection points.
- Key points such as intercepts, local maxima/minima, and asymptotes.

3. Draw a smooth curve connecting the points.

Example: Sketch the Graph of $f(x) = \frac{x^2-1}{x^2-4}$

We will sketch the graph of $f(x) = \frac{x^2-1}{x^2-4}$ by following the steps.

Step 1: Domain and Intercepts

- **Domain**: The function $f(x)$ is undefined where the denominator $x^2 - 4 = 0$. Solve:

$$x^2 - 4 = 0 \implies x = \pm 2$$

  So, the domain is $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$.

- **x-intercepts**: Solve $f(x) = 0$. The numerator $x^2 - 1 = 0$, so:

$$x^2 - 1 = 0 \implies x = \pm 1$$

  The x-intercepts are $x = 1$ and $x = -1$.

- **y-intercept**: Evaluate $f(0)$:

$$f(0) = \frac{0^2 - 1}{0^2 - 4} = \frac{-1}{-4} = \frac{1}{4}$$

  The y-intercept is $\left(0, \frac{1}{4}\right)$.

Step 2: Asymptotes

- **Vertical Asymptotes**: The denominator $x^2 - 4 = 0$ leads to vertical asymptotes at $x = \pm 2$.

- **Horizontal Asymptote**: Analyze the behavior of $f(x)$ as $x \to \pm\infty$:

$$\lim_{x \to \pm\infty} \frac{x^2 - 1}{x^2 - 4} = \frac{x^2}{x^2} = 1$$

The horizontal asymptote is $y = 1$.

Step 3: First Derivative Analysis ($f'(x)$)

1. Compute $f'(x)$ using the Quotient Rule:

$$f'(x) = \frac{(x^2 - 4)(2x) - (x^2 - 1)(2x)}{(x^2 - 4)^2}$$

   Simplify:

$$f'(x) = \frac{2x(x^2 - 4 - x^2 + 1)}{(x^2 - 4)^2} = \frac{-6x}{(x^2 - 4)^2}$$

2. **Critical Points**:
   - Solve $f'(x) = 0$:

$$\frac{-6x}{(x^2 - 4)^2} = 0 \implies x = 0$$

     So, $x = 0$ is the only critical point.

3. **Intervals of Increase/Decrease**:
   - Analyze the sign of $f'(x)$:
     - For $x > 0$, $f'(x) < 0$ (decreasing).
     - For $x < 0$, $f'(x) > 0$ (increasing).

   The function is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$.

4. **Classification of Critical Points**:
   - At $x = 0$, $f'(x)$ changes from positive to negative. Hence, $x = 0$ is a local maximum.

Step 4: Second Derivative Analysis ($f''(x)$)

1. Compute $f''(x)$:
   Differentiate $f'(x) = \frac{-6x}{(x^2-4)^2}$ using the Quotient Rule:

$$f''(x) = \frac{(x^2-4)^2(-6) - (-6x)(2)(x^2-4)(2x)}{(x^2-4)^4}$$

Simplify (tedious calculations omitted here):

$$f''(x) = \frac{-6(x^2-4)^2 + 24x^2(x^2-4)}{(x^2-4)^3}$$

2. **Concavity**:
   - Analyze the sign of $f''(x)$ to determine concavity.
   - Solve $f''(x) = 0$ to find inflection points.

Step 5: Sketch the Graph

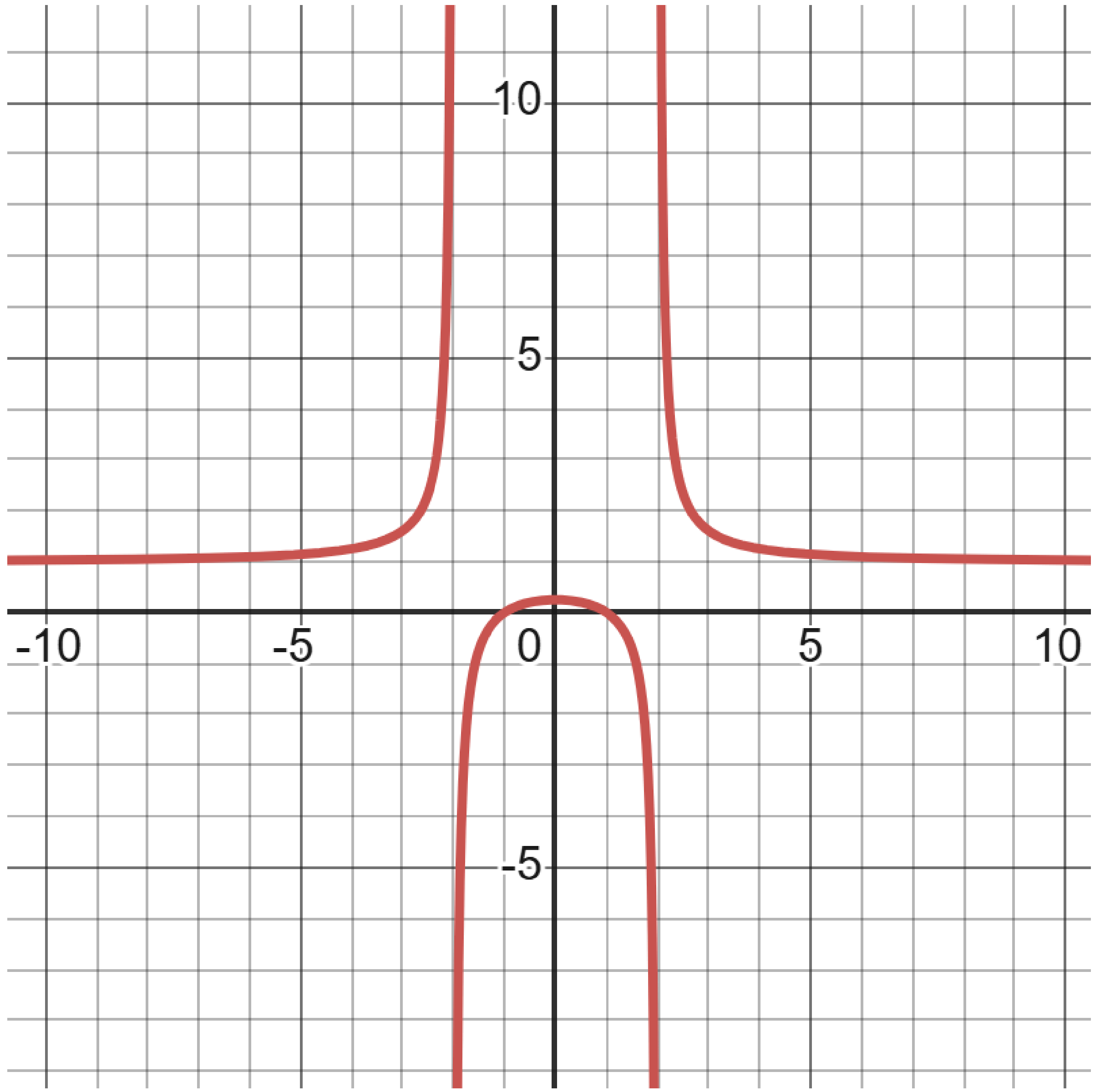
Using all the information:

- **Domain**: $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$
- **x-intercepts**: $x = -1, 1$
- **y-intercept**: $\left(0, \frac{1}{4}\right)$
- **Asymptotes**: Vertical at $x = \pm 2$, horizontal at $y = 1$
- **Critical Point**: $x = 0$ (local maximum)
- **Behavior**:
  - Increasing on $(-\infty, 0)$,
  - Decreasing on $(0, \infty)$.

Final Graph

The graph has the following features:

- Two vertical asymptotes at $x = \pm 2$,
- A horizontal asymptote at $y = 1$,
- Smooth curve connecting all points, respecting the intervals of increase/decrease and concavity.


10
5
-5
-10
-5
0
5
10

16. Integration and Riemann Sums

Connection Between Integration and Sigma Notation

Integration is closely related to summation. The definite integral of a function over an interval is the limit of the Riemann sums as the partition becomes infinitely fine. Formally:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

where:

- $[a, b]$ is the interval,
- $\Delta x = \frac{b-a}{n}$ is the width of each subinterval,
- x_i is the sample point in the i-th subinterval.

Example: Connection with Sigma Notation

Let $f(x) = x$ on $[1, 5]$. Divide $[1, 5]$ into n subintervals:

- Partition points: $x_i = 1 + i\Delta x$ where $\Delta x = \frac{5-1}{n} = \frac{4}{n}$.

Lower Sum:

$$L(f, P_n) = \sum_{i=1}^{n} f(x_{i-1})\Delta x = \sum_{i=1}^{n} \left(1 + \frac{4(i-1)}{n}\right) \frac{4}{n}.$$

Simplify:

$$L(f, P_n) = \frac{4}{n} \sum_{i=1}^{n} \left(1 + \frac{4(i-1)}{n}\right) = 4 \cdot \frac{n-1}{n} + \frac{16}{n^2} \sum_{i=1}^{n} (i-1).$$

As $n \to \infty$, $L(f, P_n) \to 12$.

Upper Sum:

$$U(f, P_n) = \sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} \left(1 + \frac{4i}{n}\right) \frac{4}{n}.$$

As $n \to \infty$, $U(f, P_n) \to 12$.

Hence, the integral is:

$$\int_1^5 x\,dx = 12.$$

Definitions

Riemann Sums

Let $f : [a, b] \to \mathbb{R}$ be a function and $P : a = x_0 < x_1 < \cdots < x_n = b$ be a partition of $[a, b]$. Define:

- **Lower Riemann Sum**:

$$L(f, P) = \sum_{i=1}^{n} f(x_i^{\min})\Delta x_i, \quad \Delta x_i = x_i - x_{i-1}.$$

- **Upper Riemann Sum**:

$$U(f, P) = \sum_{i=1}^{n} f(x_i^{\max})\Delta x_i.$$

Area

The exact area under the curve is the limit of the Riemann sums:

$$A = \lim_{n\to\infty} L(f, P_n) = \lim_{n\to\infty} U(f, P_n).$$

Properties of Riemann Sums

1. **Bounds on Area:**

$$L(f, P) \leq A \leq U(f, P).$$

2. **Refinement of Partitions**:
   If P_2 is a finer partition than P_1:

$$L(f, P_1) \leq L(f, P_2) \leq A \leq U(f, P_2) \leq U(f, P_1).$$

Definition of the Definite Integral

Let $f : [a, b] \to \mathbb{R}$. The function f is integrable on $[a, b]$ if there exists a unique real number I such that for any partition P:

$$L(f, P) \leq I \leq U(f, P).$$

This number I is called the definite integral of f over $[a, b]$, denoted as:

$$I = \int_a^b f(x)\, dx.$$

Here:

- a: Lower limit of the integral,
- b: Upper limit of the integral,
- $f(x)$: Integrand,

- dx: Integration parameter.

Example: Riemann Sum Approximation

For $f(x) = x$ on $[1, 5]$, we calculated:

$$\int_1^5 x\,dx = 12.$$

Algebraic Properties of the Definite Integral

1. **Additivity**:

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx.$$

2. **Scalar Multiplication**:

$$\int_a^b c \cdot f(x)\,dx = c\int_a^b f(x)\,dx$$

3. **Linearity**:

$$\int_a^b [cf(x) + g(x)]\,dx = c\int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

4. **Zero Interval**:

$$\int_a^a f(x)\,dx = 0.$$

5. **Order of Integration**:

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$$

6. **Comparison**:
   If $f(x) \leq g(x)$ for all $x \in [a, b]$, then:

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$$

7. **Triangle Inequality**:

$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx.$$

Remarks

- The definite integral generalizes the sum of rectangles (Riemann sums) to infinitely many subintervals.
- Finer partitions lead to more precise approximations of the integral.

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17. Integration and Fundamental Theorem of Calculus

Continuous Functions and Integrability

Theorem: Integrability of Continuous Functions

If f is continuous on $[a, b]$, then f is integrable on $[a, b]$. This means the definite integral:

$$\int_a^b f(x)\,dx$$

exists.

Average Value of a Function

Definition

For a continuous and integrable function f on $[a, b]$, the **average value** of f is defined as:

$$f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Example

Let $f(x) = x$ on $[1, 5]$. The integral is:

$$\int_1^5 x\,dx = 12.$$

The average value is:

$$f_{\text{avg}} = \frac{1}{5-1} \cdot 12 = 3.$$

Fundamental Theorem of Calculus (FTC)

Statement of FTC

Let $f : [a, b] \to \mathbb{R}$ be a continuous function. The FTC consists of two parts:

1. **Part 1 (Evaluation of Definite Integrals)**:

   If $F(x)$ is an anti-derivative of $f(x)$, meaning $F'(x) = f(x)$ for all $x \in [a, b]$, then:

   $$\int_a^b f(x)\,dx = F(b) - F(a).$$

   ### Example

   Compute $\int_1^5 x\,dx$:
   Let $F(x) = \frac{x^2}{2} + 106$. Then $F'(x) = x = f(x)$, so F is an anti-derivative of f.
   Using the FTC:

   $$\int_1^5 x\,dx = F(5) - F(1) = \left(\frac{5^2}{2} + 106\right) - \left(\frac{1^2}{2} + 106\right) = 12.$$

2. **Part 2 (Derivative of an Integral Function)**:

   Define $G(t) = \int_a^t f(x)\,dx$. Then $G'(t) = f(t)$. This means the derivative of the integral function recovers the original function f.

Example

Compute $\int_0^\pi \sin(x)\, dx$:

The anti-derivative of $\sin(x)$ is $-\cos(x)$. Thus:

$$\int_0^\pi \sin(x)\, dx = [-\cos(x)]_0^\pi = -\cos(\pi) - (-\cos(0)) = 2.$$

Properties of Anti-Derivatives

Remarks on Anti-Derivatives

1. Anti-derivative of $\cos(x)$ is $\sin(x) + C$.
2. Anti-derivative of $\sin(x)$ is $-\cos(x) + C$.
3. Anti-derivative of x^n is:

$$\frac{x^{n+1}}{n+1} + C, \quad n \neq -1.$$

Techniques for Integration

Substitution Rule

Substitution is used for integrals that involve composite functions. It is based on the chain rule for derivatives:

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x).$$

For integration:

$$\int f'(g(x))g'(x)\, dx = f(g(x)) + C.$$

Procedure

1. Let $u = g(x)$, so $du = g'(x)dx$.
2. Rewrite the integral in terms of u.

3. Integrate with respect to u.
4. Substitute $u = g(x)$ back into the result.

Example

Compute $\int_0^{\sqrt{\pi}} \cos(x^2) \cdot 2x \, dx$:

- Let $u = x^2$, so $du = 2xdx$.
- The limits change: when $x = 0, u = 0$; when $x = \sqrt{\pi}, u = \pi$.
- The integral becomes:

$$\int_0^{\pi} \cos(u) \, du = \sin(u)\Big|_0^{\pi} = \sin(\pi) - \sin(0) = 0.$$

Natural Logarithm and Exponential Functions

Definition of $\ln(x)$

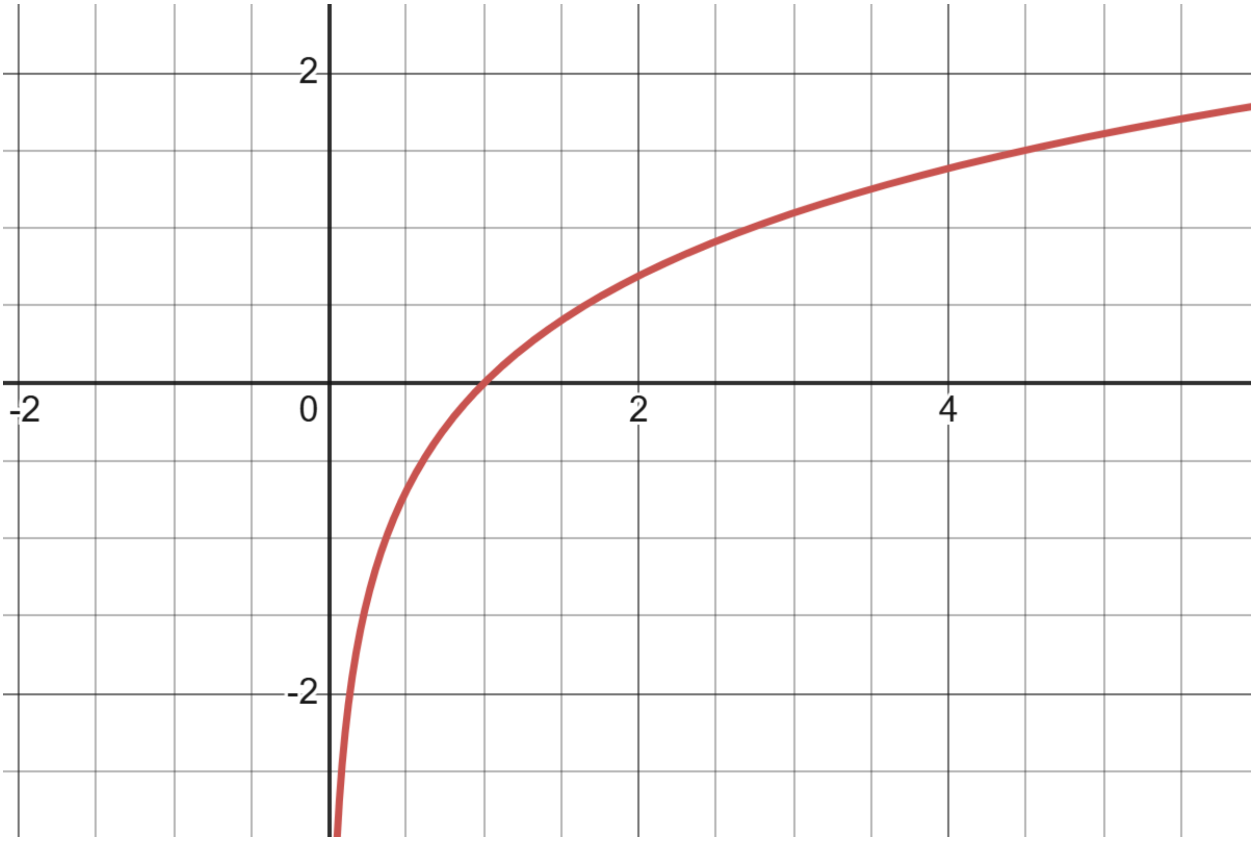
The natural logarithm is defined as:

$$\ln(x) = \int_1^x \frac{1}{t} \, dt.$$

This definition provides a way to compute $\ln(2), \ln(3)$, and other values.

Properties of $\ln(x)$

1. **Domain**: The domain of $\ln(x)$ is $(0, \infty)$.
2. **Intercept**: $\ln(1) = \int_1^1 \frac{1}{x} \, dx = 0$, so the graph passes through $(1, 0)$.
3. **Asymptotes**:
   - As $x \to 0^+$, $\ln(x) \to -\infty$ (vertical asymptote at $x = 0$).
   - As $x \to \infty$, $\ln(x) \to \infty$ (no horizontal asymptote).



Fundamental Relation Between $\ln(x)$ and e

The function $\ln(x)$ is related to the exponential function e^x as follows:

- $\ln(e) = 1$,
- e^x is the inverse of $\ln(x)$.

Exponential Function from Integration

Define $G(x) = \int_1^x \frac{1}{t}\,dt$. Then:

- $G'(x) = \frac{1}{x}$,
- $G(x)$ is increasing, continuous, and 1-to-1.

Using $G(x) = \ln(x)$, the inverse is $G^{-1}(x) = e^x$.

Example

Compute $\int_1^e \frac{1}{x}\,dx$:

$$\ln(e) - \ln(1) = 1 - 0 = 1.$$

This means the area under the curve $y = \frac{1}{x}$ from $x = 1$ to $x = e$ is exactly 1.

Historical Note: Euler and the Number e

Leonhard Euler explored the number e as the base of the natural logarithm. He showed that:

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

and that $\ln(x)$ is the integral of $\frac{1}{x}$ from 1 to x, linking e to the concept of area.

Advanced Properties of $G(x)$ and $G^{-1}(x)$

Derivatives

1. $G'(x) = \frac{1}{x} > 0$, so $G(x)$ is increasing, continuous, and bijective (1-1 and onto).
2. $G^{-1}(x) = e^x$, and:

$$(G^{-1})'(x) = \frac{1}{G'(G^{-1}(x))} = e^x.$$

Concavity

$G''(x) = -\frac{1}{x^2} < 0$, so $G(x)$ is concave down and has no inflection points.

Connection Between Natural Logarithms and Arithmetic

For $\alpha, \beta \in \operatorname{dom} G^{-1}$, we can find a, b such that $G(a) = \alpha$ and $G(b) = \beta$. Using the properties of $G(x)$:

$$G(ab) = G(a) + G(b),$$

which leads to:

$$G^{-1}(\alpha + \beta) = G^{-1}(\alpha) \cdot G^{-1}(\beta).$$

By defining $G^{-1}(1) = e$, the unique number satisfying $G(e) = 1$, we find:

$$G^{-1}(x) = e^x.$$

This means the area from 1 to α under $\frac{1}{x}$ determines $\ln(\alpha)$, and e is the number where the area is 1.

Formulas for e^x, $\ln(x)$, and $\log(x)$

e^x (Exponential Function)

Definition

$$e^x = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n$$

Derivatives

- First derivative: $(e^x)' = e^x$
- Second derivative: $(e^x)'' = e^x$
- Higher-order derivatives: $(e^x)^{(n)} = e^x$ for all $n \geq 1$

Integration

$$\int e^x \, dx = e^x + C$$

Key Properties

- $e^0 = 1$
- $e^{x+y} = e^x e^y$
- $e^{-x} = \frac{1}{e^x}$

$\ln(x)$ (Natural Logarithm)

Definition

$$\ln(x) = \int_1^x \frac{1}{t}\,dt \quad (x > 0)$$

Derivatives

- First derivative: $(\ln(x))' = \frac{1}{x}$
- Second derivative: $(\ln(x))'' = -\frac{1}{x^2}$

Integration

$$\int \ln(x)\,dx = x\ln(x) - x + C$$

Key Properties

- $\ln(1) = 0$
- $\ln(ab) = \ln(a) + \ln(b)$
- $\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$
- $\ln(a^n) = n\ln(a)$

Relationship with Exponential

- $e^{\ln(x)} = x \quad (x > 0)$
- $\ln(e^x) = x$

$\log_b(x)$ (Logarithm to Base b)

Definition

$$\log_b(x) = \frac{\ln(x)}{\ln(b)} \quad (b > 0, b \neq 1)$$

Derivatives

- First derivative: $(\log_b(x))' = \frac{1}{x\ln(b)}$

Key Properties

- $\log_b(1) = 0$
- $\log_b(b) = 1$
- $\log_b(ab) = \log_b(a) + \log_b(b)$
- $\log_b\left(\frac{a}{b}\right) = \log_b(a) - \log_b(b)$
- $\log_b(a^n) = n\log_b(a)$

Change of Base Formula

$$\log_a(x) = \frac{\log_b(x)}{\log_b(a)} \quad ,$$

for any base $b > 0, b \neq 1$

Combined Derivatives and Integrals

Exponential with Base b

- $f(x) = b^x \quad \Rightarrow \quad f'(x) = b^x \ln(b)$

$$\int b^x \, dx = \frac{b^x}{\ln(b)} + C$$

Natural Logarithm with Exponential

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax} + C \quad (a \neq 0)$$

$$(\ln(f(x)))' = \frac{f'(x)}{f(x)}$$

Logarithmic Integration

$$\int \frac{1}{x\ln(x)} \, dx = \ln(\ln(x)) + C \quad (x > 1)$$

Special Limit

$$\lim_{x \to \infty} \frac{\ln(x)}{x^a} = 0 \quad (a > 0)$$

Key Relationships

e^x and $\ln(x)$

- $\ln(e^x) = x$ and $e^{\ln(x)} = x$
- $\ln(x^a) = a \ln(x)$

Logarithmic Properties

- If $b > 1$, $\ln(x)$ and $\log_b(x)$ grow unboundedly as $x \to \infty$.

Integral Involving Logarithms

$$\int \frac{\ln(x)}{x} \, dx = \frac{\ln^2(x)}{2} + C$$

---

18. Exponential Functions and Integration by Parts

Exponential Functions of Functions

Given two functions f and g, if $f(x) > 0$ for all x in the domain of f, we define:

$$f(x)^{g(x)} = e^{\ln(f(x)^{g(x)})} = e^{g(x)\cdot\ln(f(x))}.$$

Example: Limit of $x^{1/x}$

1. Rewrite the function using the exponential property:

$$x^{1/x} = e^{\ln(x^{1/x})} = e^{\frac{\ln(x)}{x}}.$$

2. Evaluate the limit of the exponent:

$$\lim_{x\to\infty} \frac{\ln(x)}{x} = 0 \quad (\text{since } \ln(x) \text{ grows slower than } x).$$

3. Substituting back:

$$\lim_{x\to\infty} x^{1/x} = e^0 = 1.$$

Integration by Parts

Formula Derivation (Leibniz Rule)

Integration by parts is derived from the product rule for differentiation:

$$\frac{d}{dx}[u(x) \cdot v(x)] = u'(x)v(x) + u(x)v'(x).$$

Integrating both sides:

$$\int u'(x)v(x)\,dx + \int u(x)v'(x)\,dx = u(x)v(x).$$

Rearranging:

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int u'(x)v(x)\,dx.$$

This gives the integration by parts formula:

$$\int u\,dv = uv - \int v\,du.$$

Examples of Integration by Parts

$\int_0^1 xe^x\,dx$

- Let $u = x$ and $dv = e^x dx$, so $du = dx$ and $v = e^x$.
- Applying the formula:

$$\int xe^x\,dx = uv - \int v\,du = xe^x - \int e^x\,dx = xe^x - e^x + C.$$

- Evaluating from 0 to 1:

$$\int_0^1 xe^x\,dx = [1 \cdot e^1 - e^1] - [0 \cdot e^0 - e^0] = e - e + 1 = 1.$$

$\int_1^e \ln(x)\,dx$

- Let $u = \ln(x)$ and $dv = dx$, so $du = \frac{1}{x}dx$ and $v = x$.
- Applying the formula:

$$\int \ln(x)\,dx = uv - \int v\,du = x\ln(x) - \int x \cdot \frac{1}{x}\,dx = x\ln(x) - \int 1\,dx = x\ln(x) - x + C.$$

- Evaluating from 1 to e:

$$\int_1^e \ln(x)\,dx = [e\ln(e) - e] - [1\ln(1) - 1] = [e - e] - [0 - 1] = 1.$$

$\int_0^{\pi/2} \sin(x)e^x \, dx$

- Let $u = \sin(x)$, $dv = e^x dx$, so $du = \cos(x)dx$, $v = e^x$.
- Applying the formula:

$$\int \sin(x)e^x \, dx = uv - \int v \, du = \sin(x)e^x - \int \cos(x)e^x \, dx.$$

- For $\int \cos(x)e^x \, dx$, repeat integration by parts:
  - Let $u = \cos(x)$, $dv = e^x dx$, so $du = -\sin(x)dx$, $v = e^x$.

$$\int \cos(x)e^x \, dx = \cos(x)e^x - \int -\sin(x)e^x \, dx = \cos(x)e^x + \int \sin(x)e^x \, dx.$$

  - Let $I = \int \sin(x)e^x \, dx$, then:

$$I = \sin(x)e^x - (\cos(x)e^x + I) \quad \Rightarrow \quad 2I = \sin(x)e^x - \cos(x)e^x.$$

$$I = \frac{1}{2}(\sin(x)e^x - \cos(x)e^x).$$

- Evaluating from 0 to $\pi/2$:

$$\int_0^{\pi/2} \sin(x)e^x \, dx = \frac{1}{2}\left[(\sin(\pi/2)e^{\pi/2} - \cos(\pi/2)e^{\pi/2}) - (\sin(0)e^0 - \cos(0)e^0)\right].$$

$$= \frac{1}{2}\left[e^{\pi/2} - 1\right].$$

Notation and Anti-Derivatives

- **Definite Integral**:

$$\int_a^b f(x) \, dx$$

  Represents the signed area between the graph of f and the interval $[a, b]$.

- **Indefinite Integral**:

$$\int f(x) \, dx$$

  Represents an anti-derivative of f and is a function.

Example: $\int \cot(x) \, dx$

1. Rewrite using trigonometric identities:

$$\int \cot(x)\,dx = \int \frac{\cos(x)}{\sin(x)}\,dx.$$

2. Let $u = \sin(x)$, so $du = \cos(x)\,dx$:

$$\int \cot(x)\,dx = \int \frac{1}{u}\,du = \ln|u| + C = \ln|\sin(x)| + C.$$

Example: $\int \arctan(x)\,dx$

Using integration by parts:

- Let $u = \arctan(x)$, $dv = dx$, so $du = \frac{1}{1+x^2}dx$, $v = x$.
- Applying the formula:

$$\int \arctan(x)\,dx = x\arctan(x) - \int \frac{x}{1+x^2}\,dx.$$

- Simplify the remaining integral:

$$\int \frac{x}{1+x^2}\,dx = \frac{1}{2}\ln(1+x^2).$$

- Final result:

$$\int \arctan(x)\,dx = x\arctan(x) - \frac{1}{2}\ln(1+x^2) + C.$$

19. Integrals of Rational Functions and Improper Integrals

Integrals of Rational Functions

Explanation

Rational functions are expressed as the ratio of two polynomials. To integrate rational functions:

1. If the degree of the numerator is greater than or equal to the degree of the denominator, perform **polynomial division** to simplify the integrand.
2. For irreducible denominators, use **partial fraction decomposition** to break the integrand into simpler fractions.

Example 1: Polynomial Division

Evaluate:

$$\int \frac{x^2}{x-1}\,dx$$

1. Perform polynomial division:

$$\frac{x^2}{x-1} = x + 1 + \frac{1}{x-1}.$$

2. Rewrite the integral:

$$\int \frac{x^2}{x-1}\,dx = \int (x+1)\,dx + \int \frac{1}{x-1}\,dx.$$

3. Solve:

$$\int x\,dx = \frac{x^2}{2}, \quad \int 1\,dx = x, \quad \int \frac{1}{x-1}\,dx = \ln|x-1|.$$

Final result:

$$\int \frac{x^2}{x-1}\,dx = \frac{x^2}{2} + x + \ln|x-1| + C.$$

Example 2: Partial Fraction Decomposition

Evaluate:

$$\int \frac{1}{x^3+1}\,dx$$

1. Factor the denominator:

$$x^3 + 1 = (x+1)(x^2 - x + 1).$$

2. Decompose:

$$\frac{1}{x^3+1} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}.$$

3. Solve for A, B, C, and substitute into the integral:

$$\int \frac{1}{x^3+1}\,dx = \int \frac{A}{x+1}\,dx + \int \frac{Bx+C}{x^2-x+1}\,dx.$$

4. **Solve for A, B, C:**

   Multiply through by the denominator $x^3 + 1 = (x+1)(x^2 - x + 1)$ to clear the fractions:

$$1 = A(x^2 - x + 1) + (Bx + C)(x+1).$$

   Expand the right-hand side:

$$1 = A(x^2 - x + 1) + (Bx + C)(x+1) = Ax^2 - Ax + A + Bx^2 + Bx + Cx + C.$$

   Combine like terms:

$$1 = (A+B)x^2 + (-A+B+C)x + (A+C).$$

Equate coefficients with $1 = 0x^2 + 0x + 1$:

- Coefficient of x^2: $A + B = 0$
- Coefficient of x: $-A + B + C = 0$
- Constant term: $A + C = 1$

Solve the system of equations:

1. $A + B = 0 \implies B = -A$
2. $-A + (-A) + C = 0 \implies -2A + C = 0 \implies C = 2A$
3. $A + C = 1 \implies A + 2A = 1 \implies 3A = 1 \implies A = \frac{1}{3}$.

Substitute back:

- $A = \frac{1}{3}$,
- $B = -A = -\frac{1}{3}$,
- $C = 2A = \frac{2}{3}$.

5. **Rewrite the integral:**

   Substitute A, B, C into the decomposition:

$$\frac{1}{x^3+1} = \frac{\frac{1}{3}}{x+1} + \frac{-\frac{1}{3}x + \frac{2}{3}}{x^2 - x + 1}.$$

   The integral becomes:

$$\int \frac{1}{x^3+1}\,dx = \int \frac{\frac{1}{3}}{x+1}\,dx + \int \frac{-\frac{1}{3}x + \frac{2}{3}}{x^2 - x + 1}\,dx.$$

6. **Integrate each term:**

   - First term:

$$\int \frac{\frac{1}{3}}{x+1}\,dx = \frac{1}{3}\ln|x+1|.$$

   - Second term:
     Split the numerator of the second fraction:

$$\int \frac{-\frac{1}{3}x + \frac{2}{3}}{x^2 - x + 1} \, dx = -\frac{1}{3} \int \frac{x}{x^2 - x + 1} \, dx + \frac{2}{3} \int \frac{1}{x^2 - x + 1} \, dx.$$

- For $\int \frac{x}{x^2-x+1} \, dx$, use substitution $u = x^2 - x + 1$, $du = (2x - 1)dx$. This leads to a simpler integration (details skipped here for brevity).
- For $\int \frac{1}{x^2-x+1} \, dx$, complete the square:

$$x^2 - x + 1 = \left(x - \frac{1}{2}\right)^2 + \frac{3}{4}.$$

Then use a standard arctangent formula.

7. **Combine results:**

After integrating, combine all terms to express the final solution:

$$\int \frac{1}{x^3 + 1} \, dx = \frac{1}{3} \ln |x + 1| + (\text{other terms from integrations}) + C.$$

The "other terms" involve logarithms and arctangents based on the completed square and substitution.

Improper Integrals

Improper integrals involve infinite limits of integration or unbounded integrands.

Type 1: Infinite Limits

If the limits of integration include $-\infty$ or ∞:

- **Definition**:

$$\int_{-\infty}^{b} f(x) \, dx = \lim_{R \to -\infty} \int_{R}^{b} f(x) \, dx,$$

$$\int_{a}^{\infty} f(x) \, dx = \lim_{R \to \infty} \int_{a}^{R} f(x) \, dx.$$

Example 1: Convergent Integral

Evaluate:

$$\int_{-\infty}^{-1} e^x \, dx$$

1. Rewrite as a limit:

$$\int_{-\infty}^{-1} e^x \, dx = \lim_{R \to -\infty} \int_R^{-1} e^x \, dx.$$

2. Solve:

$$\int e^x \, dx = e^x + C.$$

   Substituting limits:

$$\lim_{R \to -\infty} \left[e^{-1} - e^R \right] = \frac{1}{e}.$$

The improper integral converges to $\frac{1}{e}$.

Example 2: Divergent Integral

Evaluate:

$$\int_1^{\infty} \frac{1}{x} \, dx$$

1. Rewrite as a limit:

$$\int_1^{\infty} \frac{1}{x} \, dx = \lim_{R \to \infty} \int_1^R \frac{1}{x} \, dx.$$

2. Solve:

$$\int \frac{1}{x} \, dx = \ln|x| + C.$$

   Substituting limits:

$$\lim_{R \to \infty} [\ln R - \ln 1] = \infty.$$

The improper integral diverges to ∞.

Type 2: Unbounded Integrand

If the integrand is unbounded on $(a, b]$ or $[a, b)$:

- **Definition**:

$$\int_a^b f(x)\,dx = \lim_{R\to a^+} \int_R^b f(x)\,dx,$$

$$\int_a^b f(x)\,dx = \lim_{R\to b^-} \int_a^R f(x)\,dx.$$

Example

Evaluate:

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx$$

1. Rewrite as a limit:

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{R\to 0^+} \int_R^1 x^{-1/2}\,dx.$$

2. Solve:

$$\int x^{-1/2}\,dx = 2\sqrt{x} + C.$$

   Substituting limits:

$$\lim_{R\to 0^+} \left[2\sqrt{1} - 2\sqrt{R}\right] = 2.$$

The improper integral converges to 2.

Combination of Type 1 and Type 2

Evaluate:

$$\int_{-\infty}^0 \frac{1}{x^2}\,dx$$

1. Break into two improper integrals:

$$\int_{-\infty}^{0} \frac{1}{x^2}\, dx = \int_{-\infty}^{-1} \frac{1}{x^2}\, dx + \int_{-1}^{0} \frac{1}{x^2}\, dx.$$

2. Evaluate each part using the respective definitions. Both diverge, so the integral diverges.

p-test (p-Integrals)

p-Integrals involve functions of the form $\frac{1}{x^p}$:

1. **For** $\int_1^\infty \frac{1}{x^p}\, dx$:
   - Converges if $p > 1$.
   - Diverges if $p \leq 1$.
2. **For** $\int_0^1 \frac{1}{x^p}\, dx$:
   - Converges if $p < 1$.
   - Diverges if $p \geq 1$.

Comparison Technique

Theorem

Let $0 \leq f(x) \leq g(x)$ for all $x \in [a, b]$, where a, b may be infinite.

1. If $\int_a^b g(x)\, dx$ converges, then $\int_a^b f(x)\, dx$ also converges.
2. If $\int_a^b f(x)\, dx$ diverges, then $\int_a^b g(x)\, dx$ also diverges.

Example

Compare:

$$f(x) = x^{-106}, \quad g(x) = x^{-106} + 106.$$

1. $0 \leq f(x) \leq g(x)$ for all $x \in [1, \infty)$.
2. By p-integrals, $\int_1^\infty x^{-106}\, dx$ converges.
3. By comparison, $\int_1^\infty (x^{-106} + 106)\, dx$ also converges.

20. Integrals of Rational Functions, Sequences, and Series

Integrals of Rational Functions

General Result

For the integral of a rational function $\frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials, the result typically involves:

1. **Logarithmic Terms**: $\ln|x+c|$ for linear factors in the denominator.
2. **Logarithmic and Arctangent Terms**: For irreducible quadratic factors in the denominator, the result involves $\ln|f(x)|$ or $\arctan\left(\frac{x}{\text{constant}}\right)$.

The method depends on:

1. Polynomial division (if needed),
2. Partial fraction decomposition.

Example: Improper Integral

Evaluate:

$$\int_0^\infty \frac{1}{\sqrt{x}+x^2}\,dx$$

Solution: $\int_0^\infty \frac{1}{\sqrt{x}+x^2}\,dx$

Step 1: Substitution to Simplify the Square Root

To simplify $\sqrt{x}$, substitute $x = t^2$:

- $dx = 2t\,dt$,
- $\sqrt{x} = t$, and $x^2 = t^4$.

The integral becomes:

$$\int_0^\infty \frac{1}{\sqrt{x}+x^2}\,dx = \int_0^\infty \frac{2t}{t+t^4}\,dt.$$

Step 2: Simplify the Integrand

Factor t out of the denominator:

$$\int_0^\infty \frac{2t}{t+t^4}\,dt = \int_0^\infty \frac{2t}{t(1+t^3)}\,dt.$$

Simplify further:

$$\int_0^\infty \frac{2t}{t(1+t^3)}\,dt = 2\int_0^\infty \frac{1}{1+t^3}\,dt.$$

Thus, the integral reduces to:

$$I = 2\int_0^\infty \frac{1}{1+t^3}\,dt.$$

Step 3: Split the Improper Integral

The behavior at $t = 0$ and $t \to \infty$ must be checked. The integral $\int_0^\infty \frac{1}{1+t^3}\,dt$ converges because the function decays quickly at infinity.

Step 4: Evaluate $\int_0^\infty \frac{1}{1+t^3}\,dt$

The integral of $\frac{1}{1+t^3}$ can be computed exactly using **special functions** (Beta or Gamma functions). The result is:

$$\int_0^\infty \frac{1}{1+t^3}\,dt = \frac{\pi}{3\sqrt{3}}.$$

Step 5: Substitute Back

From Step 2:

$$I = 2\int_0^\infty \frac{1}{1+t^3}\,dt.$$

Substitute the result:

$$I = 2 \cdot \frac{\pi}{3\sqrt{3}}.$$

Simplify:

$$I = \frac{2\pi}{3\sqrt{3}}.$$

Final Answer:

The value of the integral is:

$$\int_0^\infty \frac{1}{\sqrt{x}+x^2}\,dx = \frac{2\pi}{3\sqrt{3}}$$

Sequences and Series

Definition of a Sequence

A **sequence** is a function of the form:

$$a : D \to \mathbb{R},$$

where D is an infinite subset of integers, and the range is a set of real numbers.

- Notation: $a_n = a(n)$, where n is a positive integer.

Examples of Sequences

1. **Explicit Sequence**:
   Define the sequence:

$$a_n = \frac{n+1}{n^2+1}.$$

   - **Domain**: $n \in \mathbb{Z}_{\geq 0}$ (non-negative integers).
   - **Range**: $\mathbb{R}$ (real numbers).
   - The sequence maps each n to $a_n = \frac{n+1}{n^2+1}$.

2. **Logarithmic Sequence**:
   Define the sequence:

$$a_n = \ln(n), \quad n \geq 1.$$

   - **Domain**: $n \in \mathbb{Z}_{\geq 1}$ (positive integers).
   - **Range**: $\mathbb{R}$ (real numbers).
   - This sequence grows without bound as $n \to \infty$.

3. **Recursive Sequences**
   A sequence defined recursively specifies:

   a. An **initial condition** (starting value(s)),

   b. A **recurrence relation** to generate subsequent terms.

Example: Fibonacci Sequence

The Fibonacci sequence is defined as:

- **Initial conditions**: $F_0 = 1, F_1 = 1$,
- **Recursive relation**:

$$F_n = F_{n-1} + F_{n-2} \quad \text{for } n \geq 2.$$

Calculating Terms:

- $F_2 = F_1 + F_0 = 1 + 1 = 2$,
- $F_3 = F_2 + F_1 = 2 + 1 = 3$,
- $F_4 = F_3 + F_2 = 3 + 2 = 5$.

The sequence continues as: $1, 1, 2, 3, 5, 8, \ldots$.

Example: Square Root Recursive Sequence

Define the sequence:

- **Initial condition**: $a_1 = 1$,
- **Recurrence relation**:

$$a_n = \sqrt{3 + a_{n-1}} \quad \text{for } n \geq 2.$$

Calculating Terms:

- $a_1 = 1$,
- $a_2 = \sqrt{3 + a_1} = \sqrt{3 + 1} = 2$,
- $a_3 = \sqrt{3 + a_2} = \sqrt{3 + 2} = \sqrt{5}$.

The sequence continues recursively based on the relation.

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21. Solutions to Calculus Problems: Preparation for MT 2

📌 **Important Note:**

"I couldn't attend this lecture where students had an open Q&A session with the professor. To ensure I stay on track, I've included the solutions to Midterm 2 (May 4, 2015) here as a reference."

MATH 106 - Calculus I

Midterm II Solutions

Date: May 4, 2015

Problem 1 Solutions

(a) Evaluate $\lim_{x\to 0^+}\left(e^x+2x\right)^{\frac{3}{x}}$.

Solution:

1. Define the expression as $y=\left(e^x+2x\right)^{\frac{3}{x}}$.

   Take the natural logarithm on both sides to simplify the power:

$$\ln y=\frac{3}{x}\ln\left(e^x+2x\right).$$

2. The problem now reduces to finding the limit:

$$\lim_{x\to 0^+}\ln y=\lim_{x\to 0^+}\frac{3\ln\left(e^x+2x\right)}{x}.$$

3. Apply L'Hôpital's Rule:

   - The numerator is $\ln\left(e^x + 2x\right)$, and its derivative is:

$$\frac{d}{dx}\ln\left(e^x + 2x\right) = \frac{1}{e^x + 2x} \cdot \left(e^x + 2\right).$$

   - The denominator is x, and its derivative is:

$$\frac{d}{dx}x = 1.$$

   Using L'Hôpital's Rule:

$$\lim_{x \to 0^+} \frac{3\ln\left(e^x + 2x\right)}{x} = \lim_{x \to 0^+} 3 \cdot \frac{\frac{e^x+2}{e^x+2x}}{1}.$$

4. Simplify the fraction:

$$\frac{e^x + 2}{e^x + 2x} \to 1 \quad \text{as } x \to 0^+,$$

   because $e^x \to 1$ and $2x \to 0$.

5. The limit becomes:

$$\lim_{x \to 0^+} \ln y = 3 \cdot 1 = 9.$$

6. Exponentiate both sides to find y:

$$\lim_{x \to 0^+} y = e^9.$$

Final Answer: $\boxed{e^9}$

(b) Evaluate

$$\lim_{x \to 0^+} \frac{\int_{x+1}^{1-x^2} t^{106} \ln(t)\, dt}{\ln(x+1)}.$$

Solution:

1. Write the limit as:

$$\lim_{x \to 0^+} \frac{\int_{x+1}^{1-x^2} t^{106} \ln(t)\, dt}{\ln(x+1)}.$$

Both the numerator and denominator approach 0 as $x \to 0^+$, creating an indeterminate form $\frac{0}{0}$. Thus, we can apply **L'Hôpital's Rule**.

2. Differentiate the numerator using the formula for the derivative of an integral with variable limits:

$$\frac{d}{dx}\int_{x+1}^{1-x^2} t^{106}\ln(t)\,dt = t^{106}\ln(t)\Big| t = 1-x^2 \cdot \frac{d}{dx}(1-x^2) - t^{106}\ln(t)\Big| t = x+1 \cdot \frac{d}{dx}(x+1).$$

3. Compute the derivatives of the bounds:
   - At $g(x) = 1 - x^2$:

$$f(g(x)) = (1-x^2)^{106}\ln(1-x^2), \quad g'(x) = -2x.$$

   - At $h(x) = x + 1$:

$$f(h(x)) = (x+1)^{106}\ln(x+1), \quad h'(x) = 1.$$

   Substituting:

$$\frac{d}{dx}\int_{x+1}^{1-x^2} t^{106}\ln(t)\,dt = -2x(1-x^2)^{106}\ln(1-x^2) - (x+1)^{106}\ln(x+1).$$

4. Differentiate the denominator:

$$\frac{d}{dx}\ln(x+1) = \frac{1}{x+1}.$$

5. Apply L'Hôpital's Rule:

$$\lim_{x\to 0^+}\frac{\int_{x+1}^{1-x^2} t^{106}\ln(t)\,dt}{\ln(x+1)} = \lim_{x\to 0^+}\frac{-2x(1-x^2)^{106}\ln(1-x^2) - (x+1)^{106}\ln(x+1)}{\frac{1}{x+1}}.$$

6. Substitute $x = 0$ directly:

$$\lim_{x\to 0^+}\frac{-2(0)(1-)(0)^2)^{106}\ln(1-(0)^2) - ((0)+1)^{106}\ln((0)+1)}{\frac{1}{(0)+1}} = \frac{0-0}{1}$$

Thus:

$$\lim_{x\to 0^+}\frac{\int_{x+1}^{1-x^2} t^{106}\ln(t)\,dt}{\ln(x+1)} = 0$$

Final Answer: $\boxed{0}$

(c) Evaluate

$$\lim_{n \to \infty} \sum_{i=1}^{n} \frac{1}{n} \left(\frac{2i}{n} + 1 \right)^5 .$$

Solution:

1. Recognize this as a Riemann sum:

$$\lim_{n \to \infty} \sum_{i=1}^{n} \frac{1}{n} \left(\frac{2i}{n} + 1 \right)^5 .$$

   Compare it to the definition of a definite integral:

$$\int_a^b f(x)\, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f\left(x_i^*\right) \Delta x,$$

   where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$

2. Identify the corresponding integral:
   - The interval is $[0, 1]$, since $\frac{i}{n} \to x$.
   - The function is $f(x) = (2x+1)^5$.

   Thus:

$$\lim_{n \to \infty} \sum_{i=1}^{n} \frac{1}{n} \left(\frac{2i}{n} + 1 \right)^5 = \int_0^1 (2x+1)^5\, dx.$$

3. Evaluate the integral:
   Use substitution:
   - Let $u = 2x + 1$, so $du = 2\, dx$.
   - When $x = 0, u = 1$; when $x = 1$, u = 3.

   The integral becomes:

$$\int_0^1 (2x+1)^5\, dx = \frac{1}{2} \int_1^3 u^5\, du.$$

4. Compute the integral:

$$\int u^5\, du = \frac{u^6}{6}.$$

   Substituting back:

$$\frac{1}{2} \int_1^3 u^5\, du = \frac{1}{2} \left[\frac{u^6}{6} \right]_1^3 .$$

5. Evaluate the bounds:

$$\frac{1}{2}\left[\frac{3^6}{6} - \frac{1^6}{6}\right] = \frac{1}{2}\left[\frac{729}{6} - \frac{1}{6}\right] = \frac{1}{2} \cdot \frac{728}{6} = \frac{728}{12} = \frac{182}{3}.$$

Final Answer: $\boxed{\frac{182}{3}}$

Problem 2

Find the global maximum and minimum values of the function $f(x) = x^2 e^{-x^2}$.

Solution:

1. **Find the derivative:**
   To locate the critical points, compute the derivative of $f(x)$ using the product rule:

$$f'(x) = \frac{d}{dx}\left(x^2 \cdot e^{-x^2}\right) = 2xe^{-x^2} + x^2 \cdot \frac{d}{dx}\left(e^{-x^2}\right).$$

   The derivative of e^{-x^2} is:

$$\frac{d}{dx}e^{-x^2} = -2xe^{-x^2}.$$

   Substitute back:

$$f'(x) = 2xe^{-x^2} - 2x^3e^{-x^2}.$$

   Factor:

$$f'(x) = 2xe^{-x^2}(1 - x^2).$$

2. **Set $f'(x) = 0$:**
   The factors of $f'(x) = 2xe^{-x^2}(1 - x^2)$ give:

$$2x = 0 \implies x = 0, \quad 1 - x^2 = 0 \implies x = \pm 1.$$

   Critical points: $x = 0, x = 1, x = -1$.

3. **Analyze the critical points:**
   - At $x = 0$:

$$f(0) = (0)^2 e^{-(0)^2} = 0.$$

   - At $x = 1$:

$$f(1) = (1)^2 e^{-(1)^2} = e^{-1}.$$

- At $x = -1$:

$$f(-1) = (-1)^2 e^{-(-1)^2} = e^{-1}.$$

4. **Global maximum and minimum:**
    - As $x \to \pm\infty$, $f(x) = x^2 e^{-x^2} \to 0$, since the exponential decay dominates the growth of x^2.
    - The global maximum value occurs at $x = 1$ and $x = -1$, where $f(x) = e^{-1}$.
    - The global minimum value is $f(0) = 0$.

Final Answer:

- Global Maximum: $\boxed{e^{-1}}$ at $x = 1$and $x = -1$.
- Global Minimum: $\boxed{0}$ at $x = 0$.

Problem 3

(a) $\int_0^4 x^2 e^{2x}\, dx$

We will use integration by parts, which is given by:

$$\int u\, dv = uv - \int v\, du$$

Let:

- $u = x^2$, so $du = 2x\, dx$
- $dv = e^{2x}\, dx$, so $v = \frac{1}{2}e^{2x}$

Now, applying the integration by parts formula:

$$\int_0^4 x^2 e^{2x}\, dx = \left[\frac{x^2}{2}e^{2x}\right]_0^4 - \int_0^4 \frac{1}{2}e^{2x} \cdot 2x\, dx$$

Simplifying:

$$= \left[\frac{x^2}{2}e^{2x}\right]_0^4 - \int_0^4 xe^{2x}\, dx$$

Now, evaluate the boundary terms:

$$\left[\frac{x^2}{2}e^{2x}\right]_0^4 = \frac{1}{2} \cdot 16 \cdot e^8 - 0 = 8e^8$$

Thus, we now need to solve:

$$\int_0^4 x^2 e^{2x}\, dx = 8e^8 - \int_0^4 xe^{2x}\, dx$$

For the remaining integral, we apply integration by parts again on $\int_0^4 xe^{2x}\, dx$.

Let:

- $u = x$, so $du = dx$
- $dv = e^{2x}\, dx$, so $v = \frac{1}{2}e^{2x}$

Now, apply the integration by parts formula again:

$$\int_0^4 xe^{2x}\, dx = \left[\frac{x}{2}e^{2x}\right]_0^4 - \int_0^4 \frac{1}{2}e^{2x}\, dx$$

Simplifying:

$$= \left[\frac{x}{2}e^{2x}\right]_0^4 - \frac{1}{2}\int_0^4 e^{2x}\, dx$$

Evaluating the boundary terms:

$$\left[\frac{x}{2}e^{2x}\right]_0^4 = \frac{4}{2}e^8 - 0 = 2e^8$$

Now, evaluate the remaining integral:

$$\int_0^4 e^{2x}\, dx = \frac{1}{2}e^{2x}\Big|_0^4 = \frac{1}{2}\left(e^8 - 1\right)$$

Thus, we now have:

$$\int_0^4 xe^{2x}\, dx = 2e^8 - \frac{1}{2}\cdot\frac{1}{2}(e^8 - 1) = 2e^8 - \frac{1}{4}(e^8 - 1)$$

Simplifying:

$$= 2e^8 - \frac{1}{4}e^8 + \frac{1}{4} = \frac{7}{4}e^8 + \frac{1}{4}$$

Now substitute this back into the original equation for $\int_0^4 x^2 e^{2x}\, dx$:

$$\int_0^4 x^2 e^{2x}\, dx = 8e^8 - \left(\frac{7}{4}e^8 + \frac{1}{4}\right)$$

Simplifying:

$$= 8e^8 - \frac{7}{4}e^8 - \frac{1}{4} = \frac{32}{4}e^8 - \frac{7}{4}e^8 - \frac{1}{4}$$

$$= \frac{25}{4}e^8 - \frac{1}{4}$$

Thus, the final answer is:

$$\boxed{\frac{25}{4}e^8 - \frac{1}{4}}$$

(b) $\int \frac{-3\,dx}{x^3-3x^2}$

We begin by factoring the denominator:

$$x^3 - 3x^2 = x^2(x-3)$$

Now, we perform partial fraction decomposition:

$$\frac{-3}{x^2(x-3)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-3}$$

Multiplying through by the denominator $x^2(x-3)$:

$$-3 = Ax(x-3) + B(x-3) + Cx^2$$

We now find the values of A, B and C by substituting values of x:

- At $x = 0$:

$$-3 = -3B \Rightarrow B = 1$$

- At $x = 3$:

$$-3 = 3^2C \Rightarrow C = -\frac{1}{3}$$

To find A, we substitute $x = 1$:

$$A + C = 0$$

$$A = -C = \frac{1}{3}$$

Thus, the partial fraction decomposition is:

$$\frac{-3}{x^2(x-3)} = \frac{1}{3x} + \frac{1}{x^2} - \frac{1/3}{x-3}$$

Now we integrate:

$$\int \frac{-3}{x^2(x-3)}\,dx = \frac{1}{3}\int \frac{dx}{x} + \int \frac{dx}{x^2} - \frac{1}{3}\int \frac{dx}{x-3}$$

The integrals are:

$$\frac{1}{3}\ln|x| - \frac{1}{x} - \frac{1}{3}\ln|x-3| + C$$

Thus, the final answer is:

$$\frac{1}{3}\ln|x| - \frac{1}{x} - \frac{1}{3}\ln|x-3| + C$$

Problem 4

(a) Determine whether the following improper integral

$$\int_e^\infty \frac{dx}{x(\ln x)^2}$$

is convergent or divergent.

We first apply the limit as $t \to \infty$:

$$\lim_{t\to\infty} \int_e^t \frac{dx}{x(\ln x)^2}$$

Let $u = \ln x$, so $du = \frac{1}{x}dx$, thus the integral becomes:

$$= \lim_{t\to\infty} \int_{\ln e}^{\ln t} \frac{du}{u^2}$$

This simplifies to:

$$= \lim_{t\to\infty} \left[\frac{-1}{u}\right]_{\ln e}^{\ln t}$$

Evaluating the boundary terms:

$$= \lim_{t\to\infty} \left(\frac{-1}{\ln t} + 1\right)$$

As $t \to \infty$, $\frac{1}{\ln t} \to 0$, so the result is:

$$= 1$$

Since the limit exists and is finite, the integral is **convergent**.

(b) Find the area of the finite region between $y = x^3$ and $y = 3x^2 - 2x$.

The problem asks us to find the area between the curves $y = x^3$ and $y = 3x^2 - 2x$. First, we set up the integral by determining the points where the curves intersect.

Equating the two functions:

$$x^3 = 3x^2 - 2x$$

Rearranging:

$$x^3 - 3x^2 + 2x = 0$$

Factoring:

$$x(x^2 - 3x + 2) = 0$$

$$x(x-1)(x-2) = 0$$

Thus, the points of intersection are $x = 0$, $x = 1$, and $x = 2$.

Now, to find the area between the curves, we set up the integral:

$$\text{Area} = \int_0^1 \left(x^3 - 3x^2 + 2x\right) dx + \int_1^2 \left(3x^2 - 2x - x^3\right) dx$$

We now calculate each integral separately.

First integral:

$$\int_0^1 (x^3 - 3x^2 + 2x)\, dx$$

The integral is:

$$\frac{x^4}{4} - x^3 + x^2 \Big|_0^1 = \left(\frac{1}{4} - 1 + 1\right) - (0) = \frac{1}{4}$$

Second integral:

$$\int_1^2 (3x^2 - 2x - x^3)\, dx$$

The integral is:

$$\left[x^3 - x^2 - \frac{x^4}{4}\right]_1^2$$

Evaluating the bounds:

At $x = 2$:

$$2^3 - 2^2 - \frac{2^4}{4} = 8 - 4 - 4 = 0$$

At $x = 1$:

$$1^3 - 1^2 - \frac{1^4}{4} = 1 - 1 - \frac{1}{4} = -\frac{1}{4}$$

Thus, the second integral evaluates to:

$$0 - \left(-\frac{1}{4}\right) = \frac{1}{4}$$

Now, adding the results of both integrals:

$$\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

Thus, the area of the finite region is:

$$\boxed{\frac{1}{2}}$$

---

22. Sequence Properties and Mathematical Induction

Definitions

Sequence Behavior

Let $(a_n)_{n \in \mathbb{Z}^+}$ be a sequence. The following properties describe its behavior:

1. **Increasing**:

   $a_{n+1} > a_n$ for all $n \in \mathbb{Z}^+$.

2. **Decreasing**:

   $a_{n+1} < a_n$ for all $n \in \mathbb{Z}^+$.

   Example: $a_n = \frac{1}{1+n^2}$:

   - $n < n+1 \implies n^2 < (n+1)^2$ for all $n \in \mathbb{Z}^+$,
   - $n^2 + 1 < (n+1)^2 + 1$,
   - $\frac{1}{n^2+1} > \frac{1}{(n+1)^2+1}$.
     Thus, $a_n > a_{n+1}$, so a_n is **decreasing**.

3. **Non-Decreasing/Non-Increasing**:

$a_{n+1} \geq a_n$ / $a_{n+1} \leq a_n$ for all $n \in \mathbb{Z}^+$.

4. **Monotone**:

   a_n is either increasing, decreasing, non-increasing, or non-decreasing.

   Example: $a_n = \frac{1}{1+n^2}$ is monotone.

   Example: $a_n = \ln(n)$:

   - $n < n+1 \implies \ln(n) < \ln(n+1)$ because $f(x) = \ln(x)$ is increasing on $(0, \infty)$
     ($f'(x) = \frac{1}{x} > 0$ for all $x > 0$).
   - Thus, $a_n < a_{n+1}$, and a_n is **increasing**. Therefore, $\ln(n)$ is monotone.

Boundedness

1. **Bounded Above**:

   a_n is bounded above by some $M \in \mathbb{R}$ if $a_n < M$ for all $n \in \mathbb{Z}^+$.

2. **Bounded Below**:

   a_n is bounded below by some $M \in \mathbb{R}$ if $a_n > M$ for all $n \in \mathbb{Z}^+$.

   Example: $a_n = \sin(n)$:

   - $-1 \leq \sin(n) \leq 1$ for all $n \in \mathbb{Z}^+$.
     Thus, a_n is bounded above by 1 and bounded below by -1.
   - **Example**: $a_n = n^2 + 1, n \in \mathbb{Z}^+$, is bounded below by 2:

     $n \geq 1 \implies n^2 \geq 1 \implies n^2 + 1 \geq 2 > 1.$

3. **Bounded**:

   a_n is bounded if it is bounded both above and below.

   - **Example**: $a_n = \sin(n)$ is bounded.
   - **Counterexample**: $a_n = n^2 + 1$ is **unbounded** because it has no upper bound.

4. **Alternating**:

   a_n is alternating if $a_n \cdot a_{n+1} < 0$ for any n.

   Example: $a_n = (-1)^n$:

- For $n = 1, a_1 = -1$,
- For $n = 2, a_2 = 1$,
- Alternates between positive and negative.

Function Modeling Sequences

Given a sequence $(a_n)n \in \mathbb{Z}^+$*, if there exists a function* $f : \mathbb{R}_{>0} \to \mathbb{R}$ such that $f(n) = a_n$, then we say f models a_n.

Example:

Consider $a_n = \frac{\ln(n)}{n^2}, n \geq 3$. Define $f : \mathbb{R}_{\geq 3} \to \mathbb{R}$:

$$f(x) = \frac{\ln(x)}{x^2}, \quad \text{so that } f(n) = a_n.$$

- Compute $f'(x)$:

$$f'(x) = \frac{\frac{1}{x}x^2 - \ln(x) \cdot 2x}{x^4} = \frac{1 - 2\ln(x)}{x^3}.$$

- For $x \geq 3, \ln(x) \geq \ln(3) > 1 \implies 1 - 2\ln(x) < 0$.
  Thus, $f'(x) < 0$ for all $x \geq 3$.
  Since f is decreasing for $x \geq 3$, a_n is decreasing for $n \geq 3$.

Mathematical Induction

Principle of Mathematical Induction

To prove a statement P_n for all $n \in \mathbb{Z}^+$, we perform two steps:

1. **Base Case**: Show P_1 is true.
2. **Inductive Step**: Assume P_k is true for some $k \geq 1$ (inductive hypothesis). Show that $P_k \implies P_{k+1}$.

If both steps are satisfied, P_n is true for all $n \in \mathbb{Z}^+$.

Example: Prove $a_n = \sqrt{3 - a_{n-1}}$ is Increasing

Let $a_1 = 1$ and $a_n = \sqrt{3 - a_{n-1}}, n \geq 2$.

Claim: $a_n < a_{n+1}$ for all $n \in \mathbb{Z}^+$.

Proof by Mathematical Induction

1. **Base Case**:

   For $n = 1$, $a_1 = 1$ and $a_2 = \sqrt{3 - a_1} = \sqrt{3 - 1} = \sqrt{2}$.

   Since $1 < \sqrt{2}$, the base case holds.

2. **Inductive Step**:

   Assume $a_k < a_{k+1}$ for some $k \geq 1$.

   That is, $a_k < \sqrt{3 - a_k}$.

   We need to prove $a_{k+1} < a_{k+2}$, i.e., $a_{k+1} < \sqrt{3 - a_{k+1}}$.

   From the inductive hypothesis, $a_k < a_{k+1}$.

   Since a_k is increasing and $a_{k+1} < \sqrt{3 - a_{k+1}}$, it follows that $a_k < a_{k+1} < a_{k+2}$.

By the principle of mathematical induction, a_n is increasing for all $n \in \mathbb{Z}^+$.

Additional Examples

Example 1: Alternating Sequence

Let $a_n = (-1)^n$:

- $a_1 = -1, a_2 = 1, a_3 = -1, \ldots$,
- Since $a_n \cdot a_{n+1} < 0$, a_n is alternating.

Example 2: Bounded Sequence

Let $a_n = \sin(n)$:

- $-1 \leq \sin(n) \leq 1$,
- Bounded above by 1 and below by -1.

Example 3: Unbounded Sequence

Let $a_n = n^2 + 1$:

- a_n has no upper bound, so it is unbounded.

23. Sequences and Series

Definitions and Properties of Sequences

Limit of a Sequence

Let $(a_n)_{n \in \mathbb{Z}^+}$ be a sequence. We say that the **limit** of a_n is L, and write:

$$\lim_{n \to \infty} a_n = L,$$

if for every $\epsilon > 0$, there exists $N \in \mathbb{Z}^+$ such that:

$$|a_n - L| < \epsilon, \quad \forall n \geq N.$$

Remark: All standard limit rules for functions apply to sequences.

Examples:

1. $a_n = \frac{1}{n}$:

$$\lim_{n \to \infty} \frac{1}{n} = 0.$$

2. $a_n = \sin(n)$:

$$\lim_{n\to\infty} \sin(n) \text{ does not exist (d.n.e).}$$

3. $a_n = (-1)^n = \cos(\pi n)$:

$$\lim_{n\to\infty} \cos(\pi n) \text{ does not exist (d.n.e).}$$

Limit Properties for Sequences

Let $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$. Then:

1. **Sum Rule**:

$$\lim_{n\to\infty} (a_n + b_n) = L + M.$$

2. **Difference Rule**:

$$\lim_{n\to\infty} (a_n - b_n) = L - M.$$

3. **Product Rule**:

$$\lim_{n\to\infty} (a_n b_n) = L \cdot M.$$

4. **Quotient Rule** (if $M \neq 0$):

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{M}.$$

5. **Scalar Multiplication**:

$$\lim_{n\to\infty} (ca_n) = cL, \quad \text{for any constant } c.$$

Squeeze Theorem

If $a_n \leq b_n \leq c_n$ holds for all $n \in \mathbb{Z}^+$, and:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L,$$

then:

$$\lim_{n\to\infty} b_n = L.$$

Example:

Evaluate $\lim_{n\to\infty} a_n$, where $a_n = \frac{\sin^2(n)}{n^3+1}$.

1. **Bound the numerator**:

$$0 \leq \sin^2(n) \leq 1, \quad \forall n \geq 1.$$

2. **Rewrite the sequence**:

$$0 \leq \frac{\sin^2(n)}{n^3+1} \leq \frac{1}{n^3+1}.$$

3. **Apply limits to the bounds**:

$$\lim_{n\to\infty} \frac{1}{n^3+1} = 0.$$

   By the Squeeze Theorem:

$$\lim_{n\to\infty} a_n = 0.$$

Monotone Convergence Theorem (MCT)

Given a sequence $(a_n)_{n\in\mathbb{Z}^+}$:

1. If a_n is **monotone** (either increasing or decreasing), and
2. a_n is **bounded** (above or below),

then $\lim_{n\to\infty} a_n$ exists.

Example:

Consider the sequence $a_{n+1} = \sqrt{3 + a_n},\ a_1 = 1.$

1. **Monotonicity**:

   Previously shown that a_n is increasing:

$$a_n < a_{n+1} \quad \forall n \geq 1.$$

2. **Boundedness**:

   Prove by induction that $a_n \leq 6$:

   - **Base Case**: $a_1 = 1 \leq 6$.
   - **Inductive Step**: Assume $a_k \leq 6$. Show $a_{k+1} \leq 6$:

   $$a_{k+1} = \sqrt{3 + a_k} \leq \sqrt{3 + 6} = \sqrt{9} = 3 \leq 6.$$

   By induction, $a_n \leq 6$ for all n.

   Since a_n is monotone increasing and bounded above, $\lim_{n\to\infty} a_n$ exists.

3. **Find the Limit**:
   Let $\lim_{n\to\infty} a_n = \alpha$. Then:

   $$\alpha = \sqrt{3 + \alpha}.$$

   Square both sides:

   $$\alpha^2 = 3 + \alpha.$$

   Rearrange:

   $$\alpha^2 - \alpha - 3 = 0.$$

   Solve the quadratic equation:

   $$\alpha = \frac{1 \pm \sqrt{13}}{2}.$$

   Since $a_n > 0$, take the positive root:

   $$\alpha = \frac{1 + \sqrt{13}}{2}.$$

   Thus:

$$\lim_{n\to\infty} a_n = \frac{1+\sqrt{13}}{2}.$$

Subsequences

A **subsequence** of $(a_n)_{n\in\mathbb{Z}^+}$ is a sequence $(a_{k_n})_{k\in\mathbb{Z}^+}$, where k_n is a strictly increasing sequence of indices.

Remark:

If $\lim_{n\to\infty} a_n = L$, then any subsequence (a_{k_n}) also converges to L.

Example:

Let $a_n = (-1)^n \left(1 + \frac{1}{n^2}\right)$.

1. Define subsequences:
   - $a_{2n} = \left(1 + \frac{1}{(2n)^2}\right) \to 1.$
   - $a_{2n+1} = -\left(1 + \frac{1}{(2n+1)^2}\right) \to -1.$
2. Since subsequences have different limits (1 and -1), the sequence a_n does not converge:

$$\lim_{n\to\infty} a_n \text{ does not exist (d.n.e).}$$

Series

Definition of a Series

Given a sequence $(a_n)_{n\in\mathbb{Z}^+}$, *the **series** $\sum_{n=1}^{\infty} a_n$* is defined as the limit of its partial sums S_N, where:

$$S_N = \sum_{n=1}^{N} a_n = a_1 + a_2 + \cdots + a_N.$$

If $\lim_{N\to\infty} S_N = S$, we say the series **converges** to S. Otherwise, it **diverges**.

Examples:

1. $a_n = n$, $n \geq 1$:

$$S_N = \sum_{n=1}^{N} n = \frac{N(N+1)}{2}.$$

   As $N \to \infty$, $S_N \to \infty$. Thus, $\sum_{n=1}^{\infty} n$ diverges.

2. $a_n = (-1)^n$:

$$S_N = \begin{cases} -1 & \text{if } N \text{ is odd,} \\ 0 & \text{if } N \text{ is even.} \end{cases}$$

   The series does not converge, as S_N oscillates.

Summary of Key Concepts

1. **Monotone Convergence Theorem** ensures convergence of bounded monotonic sequences.
2. **Squeeze Theorem** applies to bounding sequences to find their limits.
3. A **series** converges if the limit of its partial sums exists.
4. Subsequences inherit the limits of their parent sequences but can indicate divergence when limits differ.

---

24. Sequence of Partial Sums and Convergence Tests of Series

Partial Sums and Series

Definition of a Partial Sum

> **Definition: Given a sequence (a_n) where $n \in \mathbb{Z}^+$, the N-th partial sum is defined as**
>
> $$S_N = \sum_{n=1}^{N} a_n$$
>
> **The collection $\{S_N\}_{N=1}^{\infty}$ is called the sequence of partial sums for (a_n).**

Definition of a Series

> **Definition: The series associated to the sequence (a_n) is defined as the limit of its sequence of partial sums:**

> $$\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \sum_{n=1}^{N} a_n$$
>
> **If this limit exists (and is a finite number), we say that the series converges. Otherwise, we say that the series diverges.**

Example: A Geometric-Type Sequence

Consider (a_n) defined by $a_n = A \cdot r^n$.

1. The partial sum is

$$S_N = A + Ar + Ar^2 + \cdots + Ar^N = A\frac{1 - r^{N+1}}{1 - r} \quad (\text{assuming } r \neq 1)$$

2. **Convergence**:
   - If $|r| < 1$, then $\lim_{N\to\infty} r^{N+1} = 0$, so

$$\lim_{N\to\infty} S_N = A\frac{1-0}{1-r} = \frac{A}{1-r}$$

   - If $|r| > 1$, $\lim_{N\to\infty} r^{N+1}$ does not converge to 0, and $\{S_N\}$ diverges.
   - If $r = 1$, then $S_N = A(N+1)$, which diverges as $N \to \infty$.
   - If $r = -1$, the partial sums oscillate: S_N does not settle to a single limit, so the series diverges.

Example: $\sin(n)$

If $a_n = \sin(n)$, we **do not** have a simple closed form for the partial sums, and in fact, the partial sums do not converge. Hence, $\sum_{n=1}^{\infty} \sin(n)$ diverges.

n-th Term Test for Divergence (Test for Divergence)

Statement and Proof

> **Theorem (nth Term Test):**
> **If the series $\sum_{n=1}^{\infty} a_n$ converges, then**

> $$\lim_{n\to\infty} a_n = 0$$

Proof (by contradiction / limit argument):

1. Suppose $\sum_{n=1}^{\infty} a_n$ converges to some limit L.
2. This means the partial sums $S_N = \sum_{n=1}^{N} a_n$ converge to L.
3. If $\lim_{n\to\infty} a_n \neq 0$, then eventually a_n stays away from 0, making $\{S_N\}$ fail to converge properly.
4. Therefore, a necessary condition for convergence is $\lim_{n\to\infty} a_n = 0$.

> **Remark: The converse is not true. If $\lim_{n\to\infty} a_n = 0$, it does not imply $\sum_{n=1}^{\infty} a_n$ converges. A standard counterexample is:**
>
> $$a_n = \frac{1}{n}$$
>
> **Although $\lim_{n\to\infty} \frac{1}{n} = 0$, the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.**

Contrapositive Form

> If $\lim_{n\to\infty} a_n \neq 0$ (or does not exist), then the series $\sum_{n=1}^{\infty} a_n$ diverges.

Example: $a_n = \sin(n)$. The limit $\lim_{n\to\infty} \sin(n)$ does not exist (and is not 0), so $\sum_{n=1}^{\infty} \sin(n)$ diverges by the **nth Term Test**.

Tests for Positive Series

We now focus on **positive term** series, i.e., $a_n > 0$ for all n.

1) Integral Test

> **Integral Test: Suppose $\{a_n\}$ is a positive sequence, and there is a function $f : [1, \infty) \to \mathbb{R}$ such that**
>
> 1. **f is continuous,**
> 2. **f is non-increasing,**
> 3. **$f(n) = a_n$ for all $n \in \mathbb{Z}^+$.**

> **Then the improper integral $\int_1^\infty f(x)\,dx$ and the series $\sum_{n=1}^\infty a_n$ have the same behavior: if one converges, the other converges; if one diverges, the other diverges.**

Example: $\sum_{n=1}^\infty \frac{1}{1+n^2}$. We compare it with the integral

$$\int_1^\infty \frac{dx}{1+x^2}$$

which is a convergent improper integral ($\arctan(x)$ from 1 to ∞). Thus, by the Integral Test, the series $\sum_{n=1}^\infty \frac{1}{1+n^2}$ converges.

2) p-Test (p-Series)

> **p-Test: Consider the series $\displaystyle\sum_{n=1}^\infty \frac{1}{n^p}$.**
>
> - **If $p > 1$, the series converges.**
> - **If $0 < p \le 1$, the series diverges.**

Example:

- $\sum_{n=1}^\infty \frac{1}{n^3}$ converges (here $p = 3 > 1$).
- $\sum_{n=1}^\infty \frac{1}{\sqrt{n}}$ diverges (here $p = \frac{1}{2} \le 1$).

3) Comparison Test

> **Comparison Test:**
> **Let $\{a_n\}$ and $\{b_n\}$ be sequences of positive terms with $0 \le a_n \le b_n$ for all n.**
>
> - **If $\sum b_n$ converges, then $\sum a_n$ converges.**
> - **(Contrapositively) If $\sum a_n$ diverges, then $\sum b_n$ diverges.**

Example:

$$a_n = \frac{3n+1}{n^3+7}$$

1. Note that for $n \geq 1$, $a_n > 0$.
2. Compare

$$\frac{3n+1}{n^3+7} \quad \text{with} \quad \frac{3n+1}{n^3}$$

3. For large n, $\frac{3n+1}{n^3} \sim \frac{3}{n^2} + \frac{1}{n^3}$, which converges by the p-test (since $p = 2$ or $p = 3$ both > 1). Thus $\sum_{n=1}^{\infty} a_n$ converges by the Comparison Test.

4) Limit Comparison Test

> **Limit Comparison Test:**
> **Let $\{a_n\}$ and $\{b_n\}$ be positive sequences, and consider**
>
> $$L = \lim_{n \to \infty} \frac{a_n}{b_n}$$
>
> - **If $0 < L < \infty$, then $\sum a_n$ and $\sum b_n$ either both converge or both diverge.**
> - **If $L = 0$, that means $b_n \gg a_n$. Then:**
>   - **If $\sum b_n$ converges, $\sum a_n$ converges.**
>   - **If $\sum a_n$ diverges, $\sum b_n$ diverges.**
> - **If $L = \infty$, that means $a_n \gg b_n$. Then:**
>   - **If $\sum a_n$ converges, $\sum b_n$ converges.**
>   - **If $\sum b_n$ diverges, $\sum a_n$ diverges.**

Example:

$$a_n = \frac{n^2 - 7}{n^3 + n}, \quad b_n = \frac{1}{n}$$

For large n,

$$\frac{a_n}{b_n} = \frac{\frac{n^2-7}{n^3+n}}{\frac{1}{n}} = \frac{n^2 - 7}{n^3 + n} \cdot n = \frac{n^3 - 7n}{n^3 + n}$$

As $n \to \infty$, this fraction behaves like

$$\frac{n^3}{n^3} = 1$$

so $L = 1 \in (0, \infty)$. Therefore, $\sum a_n$ converges or diverges **if and only if** $\sum b_n$ converges or diverges. But $\sum \frac{1}{n}$ diverges (harmonic series). Hence, $\sum_{n=1}^{\infty} \frac{n^2-7}{n^3+n}$ also diverges by the Limit Comparison Test.

Formulas for Summing Powers of r^n

The summation formulas for powers of r^n (geometric series) are derived as follows, assuming $|r| < 1$ (so the series converges).

1. Sum of r^n from $n = 0$ to ∞:

For the geometric series:

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + \dots$$

> **The formula is:**
>
> $$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}, \quad \text{for } |r| < 1.$$

2. Sum of r^n from $n = k$ to ∞:

For the geometric series starting at $n = k$:

$$\sum_{n=k}^{\infty} r^n = r^k + r^{k+1} + r^{k+2} + \dots$$

Factor out r^k:

$$\sum_{n=k}^{\infty} r^n = r^k \left(1 + r + r^2 + \dots\right).$$

The sum inside the parentheses is the full geometric series from $n = 0$ to ∞:

$$1 + r + r^2 + \cdots = \frac{1}{1-r}.$$

> **Thus:**
>
> $$\sum_{n=k}^{\infty} r^n = \frac{r^k}{1-r}, \quad \text{for } |r| < 1.$$

3. Sum of r^n from $n = 1$ to ∞:

For the geometric series starting at $n = 1$:

$$\sum_{n=1}^{\infty} r^n = r + r^2 + r^3 + \dots$$

Factor out r:

$$\sum_{n=1}^{\infty} r^n = r \left(1 + r + r^2 + \dots\right).$$

The sum inside the parentheses is the full geometric series from $n = 0$ to ∞:

$$1 + r + r^2 + \cdots = \frac{1}{1-r}.$$

> **Thus:**
>
> $$\sum_{n=1}^{\infty} r^n = \frac{r}{1-r}, \quad \text{for } |r| < 1.$$

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25. Series Convergence Tests and Techniques

Ratio Test

> **Let $\{a_n\}$ be a sequence of positive terms, and define**
>
> $$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} \quad (\text{where } L \in [0,\infty]).$$
>
> **Then:**
>
> - **If $L \in [0,1)$, the series $\sum a_n$ converges.**
> - **If $L \in (1,\infty)$, the series $\sum a_n$ diverges.**
> - **If $L = 1$, the Ratio Test is inconclusive.**

Example Using the Ratio Test

Consider the series

$$\sum_{n=1}^{\infty} \frac{e^n \, \pi^{2n}}{n!}.$$

Here, $a_n = \frac{e^n \pi^{2n}}{n!}$. We calculate:

$$\frac{a_{n+1}}{a_n} = \frac{e^{n+1} \pi^{2(n+1)}}{(n+1)!} \Big/ \frac{e^n \pi^{2n}}{n!} = \frac{e^{n+1}\pi^{2(n+1)}\, n!}{(n+1)!\, e^n \pi^{2n}} = \frac{e\,\pi^2}{n+1}.$$

Taking the limit as $n \to \infty$:

$$L = \lim_{n\to\infty} \frac{e\,\pi^2}{n+1} = 0.$$

Because $L = 0 < 1$, the Ratio Test tells us the series converges.

Root Test

> **Let $\{a_n\}$ be a sequence of positive terms. Define**
>
> $$L \;=\; \lim_{n\to\infty} (a_n)^{1/n} \quad (\text{where } L \in [0, \infty]).$$
>
> **Then:**
>
> - **If $L \in [0, 1)$, the series $\sum a_n$ converges.**
> - **If $L \in (1, \infty)$, the series $\sum a_n$ diverges.**
> - **If $L = 1$, the Root Test is inconclusive.**

Example Using the Root Test

Consider the series

$$\sum_{n=1}^{\infty} \frac{n^{2n}}{\pi^{2n+1}\,\pi\, 7^n}.$$

(As stated, $a_n > 0$.)

We look at $\sqrt[n]{a_n}$ and attempt to find:

$$L = \lim_{n\to\infty} \sqrt[n]{a_n}.$$

The exact details depend on how we simplify $\frac{n^{2n}}{7^n \, \pi^{2n+1} \cdots}$, but the main point is that once we compute L, we compare L with 1:

- If $L < 1$, convergent;
- If $L > 1$, divergent;
- $L = 1$, no conclusion.

Other Tests and Examples

1. **No conclusion from Ratio Test**

   Sometimes, Ratio (or Root) Test yields $L = 1$. For instance, consider

   $$\sum_{n=1}^{\infty} \frac{1}{3n+2}.$$

   Applying the Ratio Test here might give a limit of 1, so it is inconclusive. We can then try other tests (e.g., **Limit Comparison Test**, **Integral Test**, etc.).

2. **Alternating Series Test**

   > **Suppose $\{a_n\}$ is a sequence with**
   >
   > **(i) a_n is alternating (i.e., $a_n \cdot a_{n+1} < 0$ for all n),**
   >
   > **(ii) $|a_n|$ is non-increasing, and**
   >
   > **(iii) $\lim_{n\to\infty} a_n = 0$.**
   >
   > **Then the series $\sum a_n$ converges.**

   Example:

   $$\sum_{n=1}^{\infty} \frac{(-1)^n}{3n+2}.$$

   - We check it is alternating ($(-1)^n$ changes sign).
   - $\left|\frac{1}{3n+2}\right|$ is decreasing.

- $\lim_{n\to\infty} \frac{1}{3n+2} = 0$.
  By the Alternating Series Test, this series converges.

Absolute and Conditional Convergence

> **A series $\sum a_n$ is**
>
> - ***Absolutely convergent* if $\sum |a_n|$ converges.**
> - ***Conditionally convergent* if $\sum a_n$ converges but $\sum |a_n|$ diverges.**

Examples

1. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2}$
   - $\left|\frac{(-1)^n}{1+n^2}\right| = \frac{1}{1+n^2}$, which converges (it resembles a $1/n^2$ type). So the original series is **absolutely convergent**.
2. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{3n+2}$
   - By the Alternating Series Test, it converges.
   - But $\sum \left|\frac{1}{3n+2}\right|$ diverges (similar to the harmonic series).
   - Thus this series is **conditionally convergent**.

Power Series

> **A power series centered at x_0 is a series of the form**
>
> $$F(x) = \sum_{n=0}^{\infty} a_n \left(x - x_0\right)^n.$$
>
> **The set of $x \in \mathbb{R}$ for which this series converges is the *domain* of F.**

Examples of Power Series

1. $F(x) = \sum_{n=0}^{\infty} x^n$, centered at $x_0 = 0$.
2. $F(x) = \sum_{n=0}^{\infty} n\, x^n$, also centered at 0.
3. $F(x) = \sum_{n=0}^{\infty} \frac{\ln^2(n)}{n}\,(x-2)^n$, centered at $x_0 = 2$.

Domain of Convergence

Using the **Ratio Test** on

$$\left|a_{n+1}(x-x_0)^{n+1}\right| \Big/ \left|a_n(x-x_0)^n\right| = \frac{|a_{n+1}|}{|a_n|}\,\left|x-x_0\right|.$$

- If $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L$, the condition for convergence is $L \cdot |x-x_0| < 1$.

> **(i) If $L = \infty$, the only way $L\,|x-x_0| < 1$ can hold is if $x = x_0$. So the domain is $\{x_0\}$.**
>
> **(ii) If $L = 0$, then $L\,|x-x_0| = 0$ for all x, so the series converges for every $x \in \mathbb{R}$.**
>
> **(iii) If $L \in (0,\infty)$, we get $|x-x_0| < 1/L$. We must also check endpoints for convergence.**

Radius of Convergence

> **The *radius of convergence* of a power series $\sum a_n(x-x_0)^n$ is**
>
> $$R = \frac{1}{\lim_{n\to\infty}\left|\, a_{n+1}/a_n \,\right|}.$$
>
> **(If the limit is zero, $R = \infty$. If the limit is infinity, $R = 0$.)**

Example: $\sum x^n$ has $a_n = 1$. Then $\frac{|a_{n+1}|}{|a_n|} = 1$. So $L = 1$, hence $R = 1$.

Differentiation and Integration of Power Series

> **If**
>
> $$F(x) \;=\; \sum_{n=0}^{\infty} a_n \left(x - x_0\right)^n$$
>
> **is a power series with radius of convergence R, then inside its interval of convergence**
>
> **($|x - x_0| < R$):**
>
> **1. F is differentiable term by term:**
>
> $$F'(x) = \sum_{n=1}^{\infty} n\, a_n \left(x - x_0\right)^{n-1}.$$
>
> **2. F can be integrated term by term:**
>
> $$\int F(x)\, dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1} \left(x - x_0\right)^{n+1} + C.$$

Example:

- $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, for $|x| < 1$.
- Differentiate: $\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n\, x^{n-1}$.
- Integrate similarly.

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26. Solutions to Calculus Problems: Preparation for Final Exam

26. Solutions to Calculus Problems: Preparation for Final Exam

What This Note Is About

We have three problems to analyze:

1. **Integral** $\displaystyle\int_0^1 \frac{1}{\sqrt{x(1-x)}}\,dx.$
    - Investigate convergence by splitting the integral and using comparison tests.
2. **Series** $\displaystyle\sum_{n=10}^{\infty} \frac{\sin\left(\left(n+\frac{1}{2}\right)\pi\right)}{\ln\left(\ln(n)\right)}.$
    - Show how $\sin\left(\left(n+\frac{1}{2}\right)\pi\right)$ simplifies to $(-1)^n$.
    - Use absolute convergence tests (comparison with $1/n$).

- Conclude conditional convergence via the Alternating Series Test.

3. **Series** $\displaystyle\sum_{n=2}^{\infty} \frac{2}{n \ln^3(n)}$.
   - Use the Integral Test by defining $f(x) = \frac{2}{x \ln^3(x)}$.
   - Show all conditions are satisfied and evaluate the improper integral to check convergence.

Below, we provide **all solutions** with detailed checks of convergence criteria, step-by-step proofs, and tests.

Problem 1: $\displaystyle\int_0^1 \frac{1}{\sqrt{x(1-x)}}\,dx$

Objective

Determine whether the improper integral converges or diverges, and justify the result by splitting the integral and using appropriate comparison tests.

Step-by-Step Solution

We notice that the integrand $\frac{1}{\sqrt{x(1-x)}}$ becomes problematic near $x = 0$ and $x = 1$. We will split the integral into two parts and analyze each endpoint separately:

$$\int_0^1 \frac{1}{\sqrt{x(1-x)}}\,dx = \int_0^{1/2} \frac{1}{\sqrt{x(1-x)}}\,dx + \int_{1/2}^1 \frac{1}{\sqrt{x(1-x)}}\,dx.$$

Near $x = 0$

1. **Focus on** $\displaystyle\int_0^{1/2} \frac{1}{\sqrt{x(1-x)}}\,dx$**.**

   As $x \to 0^+$, $\sqrt{x(1-x)} \approx \sqrt{x}$. Hence the integrand $\frac{1}{\sqrt{x(1-x)}}$ behaves similarly to $\frac{1}{\sqrt{x}}$.

2. **Comparison**

$$\frac{1}{\sqrt{x(1-x)}} \quad \text{vs.} \quad \frac{1}{\sqrt{x}}.$$

Near $x = 0$, $\frac{1}{\sqrt{x(1-x)}} \le C \cdot \frac{1}{\sqrt{x}}$ for some constant $C > 0$.

3. **Convergence of $\int_0^{1/2} \frac{1}{\sqrt{x}}\,dx$**

   Recall

$$\int_0^{1/2} \frac{1}{\sqrt{x}}\,dx = \left[2\sqrt{x}\right]_0^{1/2} = 2\sqrt{\frac{1}{2}} - 0 < \infty.$$

   Since $\frac{1}{\sqrt{x}}$ is integrable near 0, by the Comparison Test, $\frac{1}{\sqrt{x(1-x)}}$ is also integrable near 0.

Near $x = 1$

1. **Focus on $\int_{1/2}^{1} \frac{1}{\sqrt{x(1-x)}}\,dx$.**

   As $x \to 1^-$, $\sqrt{x(1-x)} \approx \sqrt{1-x}$. Hence the integrand $\frac{1}{\sqrt{x(1-x)}}$ behaves similarly to $\frac{1}{\sqrt{1-x}}$.

2. **Comparison**

$$\frac{1}{\sqrt{x(1-x)}} \quad \text{vs.} \quad \frac{1}{\sqrt{1-x}}.$$

   Near $x = 1$, $\frac{1}{\sqrt{x(1-x)}} \le K \cdot \frac{1}{\sqrt{1-x}}$ for some constant $K > 0$.

3. **Convergence of $\int_{1/2}^{1} \frac{1}{\sqrt{1-x}}\,dx$**

   We know

$$\int_{1/2}^{1} \frac{1}{\sqrt{1-x}}\,dx = \left[-2\sqrt{1-x}\right]_{1/2}^{1} = 0 - \left(-2\sqrt{1-\tfrac{1}{2}}\right) = 2 \cdot \frac{1}{\sqrt{2}} < \infty.$$

   Hence, by the Comparison Test, $\frac{1}{\sqrt{x(1-x)}}$ is also integrable near 1.

Conclusion for Problem 1

Since the integral converges at both endpoints $x = 0$ and $x = 1$, we conclude:

$$\int_0^1 \frac{1}{\sqrt{x(1-x)}}\,dx \quad \text{is convergent.}$$

Problem 2: $\sum_{n=10}^{\infty} \frac{\sin\left(\left(n+\frac{1}{2}\right)\pi\right)}{\ln\left(\ln(n)\right)}$

Objective

Determine whether the series

$$\sum_{n=10}^{\infty} \frac{\sin\left(\left(n+\frac{1}{2}\right)\pi\right)}{\ln\left(\ln(n)\right)}$$

converges absolutely, converges conditionally, or diverges.

Step-by-Step Solution

> **Absolute convergence** of a series $\sum a_n$ means $\sum |a_n|$ converges. If $\sum |a_n|$ diverges but $\sum a_n$ converges, we say the series is **conditionally convergent**.

Simplifying the Terms

First, observe:

$$\sin\left(\left(n+\tfrac{1}{2}\right)\pi\right) = \sin\left(n\pi + \tfrac{\pi}{2}\right) = \cos(n\pi) = (-1)^n.$$

Hence each term is:

$$a_n = \frac{(-1)^n}{\ln(\ln(n))}.$$

Absolute Convergence Test

Consider

$$\sum_{n=10}^{\infty} |a_n| = \sum_{n=10}^{\infty} \frac{1}{\ln(\ln(n))}.$$

We compare $\frac{1}{\ln(\ln(n))}$ to $\frac{1}{n}$:

1. **Inequality**

   For $n \geq 10$, we typically have

   $$\ln(\ln(n)) < \ln(n) < n \quad \implies \quad \frac{1}{\ln(\ln(n))} > \frac{1}{n}.$$

2. **Divergent Comparison**

   Since $\sum_{n=10}^{\infty} \frac{1}{n}$ diverges (harmonic series), and $\frac{1}{\ln(\ln(n))} \geq C \cdot \frac{1}{n}$ for large n, then by Comparison Test:

   $$\sum_{n=10}^{\infty} \frac{1}{\ln(\ln(n))} \quad \text{also diverges.}$$

   Therefore, the series $\sum |a_n|$ does **not** converge.

Conditional Convergence via Alternating Series Test

Now check the original (non-absolute) series:

$$\sum_{n=10}^{\infty} \frac{(-1)^n}{\ln(\ln(n))}.$$

1. **Alternating terms**: $(-1)^n$ provides the sign changes.

2. **Magnitude is decreasing**:

   $$b_n = \frac{1}{\ln(\ln(n))}.$$

   As $n \to \infty$, $\ln(\ln(n)) \to \infty$, so b_n decreases to 0.

3. **Limit to zero**:

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\ln(\ln(n))} = 0.$$

By the **Alternating Series Test** (Leibniz Test), an alternating series with terms that decrease in absolute value to 0 converges.

Conclusion for Problem 2

- **Absolute convergence**: Fails (the series $\sum |a_n|$ diverges).
- **Conditional convergence**: Succeeds, by the Alternating Series Test.

Hence, the series

$$\sum_{n=10}^{\infty} \frac{(-1)^n}{\ln(\ln(n))}$$

is **conditionally convergent**.

Problem 3: $\sum_{n=2}^{\infty} \frac{2}{n\ \ln^3(n)}$

Objective

Determine whether the series

$$\sum_{n=2}^{\infty} \frac{2}{n\ \ln^3(n)}$$

converges or diverges by using the Integral Test.

Step-by-Step Solution

> **Theorem (Integral Test)**
>
> **Suppose $f(x)$ is continuous, positive, and decreasing for $x \geq x_0$. Then the series $\sum_{n=x_0}^{\infty} f(n)$ converges if and only if the integral $\displaystyle\int_{x_0}^{\infty} f(x)\,dx$ converges.**

1. **Define**

$$f(x) = \frac{2}{x\ \ln^3(x)}, \quad x \geq 2.$$

- **Continuous**: f is continuous for $x > 1$, so certainly on $[2, \infty)$.
- **Positive**: $f(x) > 0$ for $x > 1$.
- **Decreasing**: As x increases, $\ln(x)$ grows, so $\ln^3(x)$ grows, making $\frac{2}{x \ln^3(x)}$ decrease.

2. **Evaluate** $\displaystyle\int_2^\infty \frac{2}{x\ \ln^3(x)}\, dx.$

   Let $u = \ln(x)$. Then $du = \frac{1}{x}\, dx$. When $x = 2$, $u = \ln(2)$. When $x \to \infty$, $u \to \infty$. Thus the integral becomes

$$\int_2^\infty \frac{2}{x\ \ln^3(x)}\, dx = \int_{\ln(2)}^\infty \frac{2}{u^3}\, du = 2\int_{\ln(2)}^\infty u^{-3}\, du.$$

3. **Compute the improper integral**

$$2\int_{\ln(2)}^\infty u^{-3}\, du = 2\left[-\tfrac{1}{2u^2}\right]_{\ln(2)}^\infty = 2\left(0 - \left(-\tfrac{1}{2(\ln(2))^2}\right)\right) = \frac{1}{(\ln(2))^2} < \infty.$$

   Hence the integral converges.

4. **Conclusion by Integral Test**

   Since $\displaystyle\int_2^\infty \frac{2}{x \ln^3(x)}\, dx$ converges and f satisfies the conditions of the Integral Test, the series

$$\sum_{n=2}^\infty \frac{2}{n\ \ln^3(n)}$$

   converges.

Final Summary

- **Problem 1**: The integral $\int_0^1 \frac{1}{\sqrt{x(1-x)}}\,dx$ converges by splitting into two intervals $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$ and comparing with $\frac{1}{\sqrt{x}}$ and $\frac{1}{\sqrt{1-x}}$ near the endpoints.
- **Problem 2**: The series $\sum_{n=10}^{\infty} \dfrac{(-1)^n}{\ln(\ln(n))}$ converges **conditionally** by the Alternating Series Test, but fails absolute convergence since $\sum \dfrac{1}{\ln(\ln(n))}$ diverges by comparison with $\frac{1}{n}$.
- **Problem 3**: The series $\sum_{n=2}^{\infty} \dfrac{2}{n\,\ln^3(n)}$ converges by the Integral Test, as the corresponding improper integral from 2 to ∞ converges.