# MATH107

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$$\begin{bmatrix} -3 & 4 & 0 & 2 \\ 6 & 1 & 1 & 5 \\ 5 & 10 & 8 & -2 \\ 2 & 5 & 10 & 8 \end{bmatrix}$$

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# MATH107

# 1. Linear Equations and Their Applications

## Linear Equation Definition

A **linear equation** is an equation of the form:

$$a_1x_1 + a_2x_2 + a_3x_3 + \ldots + a_nx_n = b$$

where:

- $a_i$ (coefficients) are constants,
- $x_i$ are variables, and
- $b$ is a constant.

### Example of Linear Equations

- The equation $-x_1 + 2x_2 + x_3 = 10$ is linear, representing a plane in three-dimensional space.
- The equation $x_1 + x_2x_3 = 7$ is **not linear** due to the product of variables.

### System of Linear Equations

A **system of linear equations** consists of multiple linear equations involving the same set of variables. It can be expressed in the form:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m
\end{aligned}
$$

where:

- $m$ is the number of equations,
- $n$ is the number of variables.

### Example of Solving a System of Linear Equations

Consider the system:

$$
\begin{aligned}
x_1 + 2x_2 = 3 \\
4x_1 + 5x_2 = 6
\end{aligned}
$$

Using methods such as elimination or substitution, we can find the values of $x_1$ and $x_2$.

### Possibilities for Solutions

Given a system of linear equations, there are three possible scenarios:

1. **Unique Solution**: There is exactly one solution to the system.
2. **Infinitely Many Solutions**: There are multiple solutions satisfying the equations.
3. **No Solution**: The equations are inconsistent.

- Cases with unique and infinitely many solutions are termed **consistent**.
- A system with no solution is termed **inconsistent**.

## Example of Linear Equations

- **Unique Solution**:

$$x_1 + x_2 = 5$$
$$2x_1 + 3x_2 = 12$$

- **Infinitely Many Solutions**:

$$x_1 + x_2 = 5$$
$$2x_1 + 2x_2 = 10$$

## Matrix Representation

A system of linear equations can also be represented in **matrix form**. The general form is:

$$A_{m \times n}\mathbf{x} = \mathbf{b}$$

where:

- $A$ is the matrix of coefficients,
- $\mathbf{x}$ is the vector of variables,
- $\mathbf{b}$ is the vector of constants.

### Notation

- The size of matrix $A$ is $m \times n$.
- Each element of the matrix is denoted as $a_{ij}$, where $i$ is the row index and $j$ is the column index.
- The first element $a_{11}$ refers to the value in the first row and first column.

### Square Matrix

A **square matrix** is defined as:

$$A_{n \times n}$$

where the number of rows $m$ equals the number of columns $n$.

### Elementary Row Operations

1. **(Replacement)** Replace one row by the sum of itself and a multiple of another row.
2. **(Interchange)** Interchange two rows.
3. **(Scaling)** Multiply all entries in a row by a nonzero constant.

# 2. Echelon Form and Reduced Echelon Form

## Echelon Form

A matrix $A_{m \times n}$ is in **echelon form** if it satisfies the following conditions:

1. **Nonzero Rows Above Zero Rows**: All nonzero rows appear above any rows of all zeros.
2. **Leading Entry in Each Row**: Each leading entry of a row (the first non-zero entry in a row) is in a column to the right of the leading entry of the row above it.
    - **Example**: In the matrix below, 4 is the leading entry in row 1, and 5 is the leading entry in row 2, positioned to the right of 4.
3. **Zeros Below Leading Entry**: All entries below a leading entry in a column should be zero.

### Example of a Matrix in Echelon Form:

Here, the leading entries are 4 (row 1), 5 (row 2), and 3 (row 3), with zeros below the leading entries in each column.

$$\begin{pmatrix} 4 & 5 & 6 \\ 0 & 5 & 8 \\ 0 & 0 & 3 \end{pmatrix}$$

## Reduced Echelon Form

A matrix is in **reduced echelon form** if it satisfies all the conditions of echelon form, plus the following:

1. **Leading Entry Equals 1**: Each leading entry in a row must be 1 (this is called a **leading 1**).
2. **Zeros in Leading Entry Column**: Each leading 1 is the only non-zero entry in its column (i.e., all other entries in the same column should be zero).

### Example of a Matrix in Reduced Echelon Form:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

## Converting to Reduced Echelon Form

### Algorithm (Gaussian Elimination):

1. **Find the leading entry** (non-zero element) in the first row.
2. **Make it a 1** by dividing the row by the leading entry value.
3. **Eliminate non-zero entries below this leading 1** by subtracting multiples of the first row from the rows below.
4. **Repeat for each row**: Move to the next row and apply steps 1-3 until all rows have leading 1s and zeros below and above the leading 1s.

### Example:

Consider the matrix:

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 3 & 6 & 3 \end{pmatrix}$$

We will convert this matrix into reduced echelon form.

**Step 1**: Make the first entry of row 1 a leading 1 (it already is, so no change is needed):

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 3 & 6 & 3 \end{pmatrix}$$

**Step 2**: Subtract 2 times row 1 from row 2 to eliminate the leading entry below row 1.

$$\text{Row } 2 \leftarrow \text{Row } 2 - 2 \times \text{Row } 1$$

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 3 & 6 & 3 \end{pmatrix}$$

**Step 3**: Subtract 3 times row 1 from row 3 to eliminate the leading entry below row 1.

$$\text{Row } 3 \leftarrow \text{Row } 3 - 3 \times \text{Row } 1$$

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

**Step 4**: We now have the matrix in echelon form. Since the second and third rows are zero rows, the matrix is already in reduced row echelon form (RREF):

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

This is the final solution. The matrix represents a system of linear equations where one equation has an infinite number of solutions, and the other two are dependent on the first.

# Linear Equations to Matrix Notation

Consider a system of linear equations:

$$\begin{aligned} 2x_1 + 3x_2 &= 5 \\ 4x_1 + 6x_2 &= 10 \end{aligned}$$

This system can be written in matrix form as:

$$A\mathbf{x} = \mathbf{b}$$

Where:

$$A = \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 5 \\ 10 \end{pmatrix}$$

# Augmented Matrix

The augmented matrix represents the system by combining the coefficient matrix and the constants from the right-hand side of the equations:

$$\left(\begin{array}{cc|c} 2 & 3 & 5 \\ 4 & 6 & 10 \end{array}\right)$$

# Elementary Row Operations

To solve a system of linear equations using matrices, we use **elementary row operations**:

1. **Multiply a row by a non-zero constant**.
2. **Add or subtract a multiple of one row to/from another row**.
3. **Interchange two rows**.

## Result

By using these operations, the matrix is transformed into reduced echelon form, from which the solutions to the system can be read directly.

### Theorem 1: Uniqueness of the Reduced Echelon Form

Each matrix is **row equivalent** to **one and only one reduced row echelon matrix (RREF)**.

### Definition: Pivot Position & Pivot Column

A **pivot position** in a matrix $A$ is a location in $A$ that corresponds to a leading 1 in the reduced echelon form of $A$. A **pivot column** is a column of $A$ that contains a pivot position.

## Theorem 2: Existence and Uniqueness Theorem

A linear system is **consistent** if and only if the rightmost column of the augmented matrix is **not** a pivot column—that is, if and only if an echelon form of the augmented matrix has **no row** of the form:

$$\begin{bmatrix} 0 & 0 & \dots & 0 \mid b \end{bmatrix}$$

with $b$ non-zero.

### Implications

If a linear system is consistent, then the solution set contains either:

1. **A unique solution**, when there are no free variables, or
2. **Infinitely many solutions**, when there is at least one free variable.

## Using Row Reduction to Solve a Linear System

1. **Write the augmented matrix of the system.**
    - Start by constructing the augmented matrix from the given system of linear equations.
2. **Use the row reduction algorithm** to obtain an equivalent augmented matrix in **echelon form**.
    - Perform elementary row operations to bring the matrix to row echelon form (REF).

- Decide whether the system is **consistent**.
- If there is **no solution**, stop; otherwise, proceed to the next step.

3. **Continue row reduction** to obtain the **reduced echelon form** (RREF).
   - Apply further row operations to get the matrix into reduced row echelon form.
4. **Write the system of equations** corresponding to the matrix obtained in step 3.
   - Translate the RREF matrix back into a system of equations.
5. **Rewrite each nonzero equation** from step 4 so that its one **basic variable** is expressed in terms of any **free variables** appearing in the equation.
   - Rearrange the equations to express the basic variables as functions of the free variables.

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# 3. Introduction to Vectors

## Vector Form and Notation

A vector is an ordered list of numbers, representing quantities with both magnitude and direction. For instance, a vector in $\mathbb{R}^n$ (n-dimensional real space) can be written as:

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

**Not a vector form:** Any set that does not preserve both magnitude and direction or violates the ordered structure, such as a scalar or a matrix, is not considered a vector.

## Denoting Vectors

Vectors are typically denoted in boldface (e.g., $\mathbf{v}$) or with an arrow above the letter (e.g., $\vec{v}$ ). In handwritten work, underlining or overlining can also be used to denote vectors.

## Unit Vectors in $\mathbb{R}^3$

In a 3-dimensional coordinate system, the unit vectors $\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}$ are often denoted as:

- $\mathbf{i}$ for the unit vector along the x-axis.
- $\mathbf{j}$ for the unit vector along the y-axis.
- $\mathbf{k}$ for the unit vector along the z-axis.

## Vector Addition and Multiplication by a Constant

- **Vector addition:** The sum of two vectors $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$ is:

$$\mathbf{u} + \mathbf{v} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{pmatrix}$$

- **Multiplication by a scalar:** Given a vector $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$ and a scalar $c \in \mathbb{R}$, the scalar multiplication is:

$$c\mathbf{u} = \begin{pmatrix} cu_1 \\ cu_2 \\ \vdots \\ cu_n \end{pmatrix}$$

## Zero Vector in $\mathbb{R}^n$

The zero vector $\mathbf{0} \in \mathbb{R}^n$ is the vector where all components are zero:

$$\mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

## Vector Equality

Two vectors $\mathbf{u}$ and $\mathbf{v}$ are equal if and only if all their corresponding components are equal:

$$\mathbf{u} = \mathbf{v} \quad \text{if and only if} \quad u_i = v_i \text{ for all } i$$

**Example of equality:**

$$\begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}$$

## Dimensionality

- Vectors in $\mathbb{R}^n$ and $\mathbb{R}^m$ are not equal if $n \neq m$, since they have different numbers of components.

## Properties of Vectors

Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$:

1. **Addition with zero vector:** $\mathbf{u} + \mathbf{0} = \mathbf{u}$
2. **Commutativity of addition:** $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3. **Associativity of addition:** $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
4. **Inverse of addition:** $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
5. **Distributive property:** $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$

## Transposing a Vector

The **transpose** of a vector turns a column vector into a row vector and vice versa. If $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$, then the transpose of $\mathbf{v}$ is:

$$\mathbf{v}^T = \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix}$$

## Linear Combination of Vectors in $\mathbb{R}^n$

A vector $\mathbf{v}$ is a linear combination of vectors $\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_k} \in \mathbb{R}^n$ if it can be written as:

$$\mathbf{v} = c_1\mathbf{u_1} + c_2\mathbf{u_2} + \cdots + c_k\mathbf{u_k}$$

where $c_1, c_2, \ldots, c_k \in \mathbb{R}$ are scalar coefficients.

## Span

The span of a set of vectors$\{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_k}\}$ is the set of all possible linear combinations of these vectors. In other words, the span is the smallest subspace that contains all the vectors.

### Example

Let $S = \{(1, 2), (-1, 5)\}$ and consider $(x_1, x_2) \in \mathbb{R}^2$. Can we write:

$$(x_1, x_2) = c_1(1, 2) + c_2(-1, 5)$$

for some $c_1, c_2 \in \mathbb{R}$.

By solving the system of linear equations, we can find $c_1$ and $c_2$ for any $(x_1, x_2)$, showing whether $\mathbb{R}^2 \leq S$.

## True or False Question

Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$. Is it true that $\mathrm{Span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\} = \mathbb{R}^3$?

The span equals $\mathbb{R}^3$ if and only if the vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly independent and can form a basis for $\mathbf{u}, \mathbf{v}, \mathbf{w}$.

# Vector Equation and Span

A vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

has the **same solution set** as the linear system whose augmented matrix is:

$$[\mathbf{a}_1\ \mathbf{a}_2\ \ldots\ \mathbf{a}_n \mid \mathbf{b}]$$

In particular, $\mathbf{b}$ can be **generated** by a linear combination of $\mathbf{a}_1, \ldots, \mathbf{a}_n$ if and only if there exists a solution to the linear system corresponding to the matrix above.

## Key Idea

One of the key ideas in linear algebra is to study the set of all vectors that can be generated or written as a **linear combination** of a fixed set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ of vectors.

# Definition: Span

If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are in $\mathbb{R}^n$, then the set of all linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_p$ is denoted by $Span\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ and is called the **subset** of $\mathbb{R}^n$ **spanned** (or generated) by $\mathbf{v}_1, \ldots, \mathbf{v}_p$. That is,

$$\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$$

is the collection of all vectors that can be written in the form:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p$$

with $c_1, \ldots, c_p$ as scalars.

## Relationship to the Vector Equation

Asking whether a vector $\mathbf{b}$ is in $Span\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ amounts to asking whether the vector equation:

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{b}$$

has a solution, or, equivalently, asking whether the linear system with augmented matrix:

$$\left[\mathbf{v}_1\;\mathbf{v}_2\;\ldots\;\mathbf{v}_p\;\middle|\;\mathbf{b}\right]$$

has a solution.

## Special Case: Scalar Multiples and the Zero Vector

Note that $Span\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ contains every scalar multiple of $\mathbf{v}_1$ (for example), since:

$$c\mathbf{v}_1 = c\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_p$$

In particular, the **zero vector** must be in $Span\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

# 4. Identity Matrices, Linear Combinations, and Consistency of Systems

## Identity Matrices

### Definition

An **identity matrix** is a square matrix where all diagonal elements are 1 and all other elements are 0. It is denoted by $I_n$ , where $n$ is the dimension of the matrix. For example:

$$I_n = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

### Identity Matrices vs. Unit Vectors

- The **columns** of an identity matrix are called **standard basis vectors** or **unit vectors** in $\mathbb{R}^n$.
- In physics, **unit vectors** indicate direction and have a magnitude of 1. In $\mathbb{R}^3$, the unit vectors are denoted by **i**, **j**, and **k**.

### Examples of Identity Matrices

1. $2 \times 2$ identity matrix:

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

2. $3 \times 3$ identity matrix:

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

## Theorems on Linear Combinations and Consistency

Let $A_{m \times n}$ be a matrix. The following statements are equivalent:

1. Every vector $b \in \mathbb{R}^m$ is a linear combination of the columns of $A$ .
2. For each $b \in \mathbb{R}^m$, the equation $Ax = b$ has a solution (i.e., it is consistent).
3. The columns of $A$ span $\mathbb{R}^m$.
4. $A$ has a pivot position in every row.

These statements mean that if one is true, all are true, ensuring that the columns of A can represent all vectors in $\mathbb{R}^m$.

## Example: Consistency of the System $Ax = b$

Given:

$$A = \begin{pmatrix} 1 & 2 & 0 \\ -1 & -3 & 1 \\ 3 & 7 & -1 \end{pmatrix}$$

We want to determine if the system $Ax = b$ is consistent for all $b \in \mathbb{R}^3$.

### Row Operations

Perform row operations to check for leading entries:

1. Start with the augmented matrix:

$$\left( \begin{array}{ccc|c} 1 & 2 & 0 & b_1 \\ -1 & -3 & 1 & b_2 \\ 3 & 7 & -1 & b_3 \end{array} \right)$$

2. Row operations:

   - $R_2 \to R_2 + R_1$:

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & b_1 \\ 0 & -1 & 1 & b_2 + b_1 \\ 3 & 7 & -1 & b_3 \end{array}\right)$$

   - $R_3 \to R_3 - 3R_1$:

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & b_1 \\ 0 & -1 & 1 & b_2 + b_1 \\ 0 & 1 & -1 & b_3 - 3b_1 \end{array}\right)$$

   - $R_3 \to R_3 + R_2$:

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & b_1 \\ 0 & -1 & 1 & b_2 + b_1 \\ 0 & 0 & 0 & b_3 - 2b_1 - b_2 \end{array}\right)$$

### Result

There are only **two leading entries**, so there is no pivot in the third row. Therefore, according to the theorem, the system $Ax = b$ is not consistent for all $b \in \mathbb{R}^3$.

## Matrix Multiplication Properties

For a matrix $A_{m\times n}$ and vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$:

1. **Distributive Property**:
   $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$
2. **Scalar Multiplication**:
   $A(c \cdot \mathbf{u}) = c \cdot (A\mathbf{u})$

## Example of Consistency: True or False?

**Statement**: If A is a $2 \times 3$ matrix, then $Ax = b$ is consistent for all $b \in \mathbb{R}^2$.

**Answer**: False.

### Explanation

A $2 \times 3$ matrix has only **2 rows**, so it cannot have a pivot in every row and column simultaneously. As a result, it may not be able to represent all vectors in $\mathbb{R}^2$, meaning there could be some vectors $b$ for which $Ax = b$ has no solution.

## Theorem 3

If $A$ is an $m \times n$ matrix, with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$, and if $\mathbf{b}$ is in $\mathbb{R}^m$, the matrix equation

$$A\mathbf{x} = \mathbf{b} \tag{1}$$

has the same solution set as the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b} \tag{2}$$

which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$[\mathbf{a}_1 \ \mathbf{a}_2 \ \cdots \ \mathbf{a}_n \mid \mathbf{b}] \tag{3}$$

## Theorem 4

Let $A$ be an $m \times n$ matrix. The following statements are **logically equivalent**. That is, for a particular $A$, either they are all true statements or they are all false:

- **(a)** For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.
- **(b)** Each $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of $A$.
- **(c)** The columns of $A$ span $\mathbb{R}^m$.
- **(d)** $A$ has a pivot position in every row.

## Theorem 5

If $A$ is an $m \times n$ matrix, $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$, and $c$ is a scalar, then:

- **(a)** $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$
- **(b)** $A(c\mathbf{u}) = c(A\mathbf{u})$

# Matrix-Vector Multiplication Rule

When multiplying a matrix $A$ by a vector $\mathbf{x}$, we calculate each entry of the resulting vector as the **dot product** of the rows of $A$ with the vector $\mathbf{x}$.

## Matrix-Vector Multiplication Formula

For a matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and a vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, the product $A\mathbf{x}$ is:

$$A\mathbf{x} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_{11} \cdot x_1 + a_{12} \cdot x_2 \\ a_{21} \cdot x_1 + a_{22} \cdot x_2 \end{bmatrix}$$

Each element in the resulting vector is the **sum of products** of elements in the row with corresponding elements in the vector.

## Example

Let $A = \begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

1. **Multiply the first row** by $\mathbf{x}$:

$$2 \cdot 1 + 5 \cdot 4 = 2 + 20 = 22$$

1. **Multiply the second row** by $\mathbf{x}$:

$$3 \cdot 1 + 1 \cdot 4 = 3 + 4 = 7$$

Thus, $A\mathbf{x} = \begin{bmatrix} 22 \\ 7 \end{bmatrix}$.

## Condition for Solution: Matrix Equation $A\mathbf{x} = \mathbf{b}$

The equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is a linear combination of the columns of $A$.

## Explanation

- **Matrix Equation**: The equation $A\mathbf{x} = \mathbf{b}$ represents a system of linear equations, where $A$ is a matrix, $\mathbf{x}$ is a vector of unknowns, and $\mathbf{b}$ is a result vector.

- **Linear Combination Requirement**: For $\mathbf{b}$ to be expressible as $A\mathbf{x}$, it must be possible to write $\mathbf{b}$ as a linear combination of the columns of $A$. This means there exist scalars $x_1, x_2, \ldots, x_n$ such that:

$$\mathbf{b} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n$$

where $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ are the columns of $A$.

## Key Takeaway

If $\mathbf{b}$ is **not** in the span of the columns of $A$, then the system $A\mathbf{x} = \mathbf{b}$ has **no solution**.

# 5. Homogeneous and Nonhomogeneous Linear Systems, and Linear Independence

## Homogeneous Linear System

### Definition

A **homogeneous linear system** has the form:

$$Ax = 0$$

where $A$ is a matrix and $x$ is the vector of variables. The vector $x = 0$ is always a solution to this system, called the **trivial solution**.

### Solutions of Homogeneous Systems

- **Trivial Solution**: The zero vector $x = 0$ is always a solution.

- **Non-Trivial Solutions**: Solutions other than the zero vector are called non-trivial solutions. A homogeneous system has non-trivial solutions if and only if the system has **infinitely many solutions**, which occurs when the matrix $A$ has **fewer pivots than the number of variables**.

### Solution Set

The solution set of a homogeneous system $Ax = 0$ is expressed as:

$$\text{Span}\{v_1, \ldots, v_k\}$$

where $v_1, \ldots, v_k$ are vectors in $\mathbb{R}^n$.

### Example: Homogeneous System

Consider the homogeneous system:

$$\begin{aligned} 3x_1 + 5x_2 + 4x_3 &= 0 \\ -3x_1 - 2x_2 + 4x_3 &= 0 \\ 6x_1 + x_2 - 8x_3 &= 0 \end{aligned}$$

### Step 1: Create the Augmented Matrix

$$\left(\begin{array}{ccc|c} 3 & 5 & 4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{array}\right)$$

### Step 2: Row Operations to Echelon Form

- $R_2 \to R_2 + R_1$ :

$$\left(\begin{array}{ccc|c} 3 & 5 & 4 & 0 \\ 0 & 3 & 8 & 0 \\ 6 & 1 & -8 & 0 \end{array}\right)$$

- $R_3 \to R_3 - 2R_1$ :

$$\left(\begin{array}{ccc|c} 3 & 5 & 4 & 0 \\ 0 & 3 & 8 & 0 \\ 0 & -9 & -24 & 0 \end{array}\right)$$

- $R_3 \to R_3 + 3R_2$ :

$$\left(\begin{array}{ccc|c} 3 & 5 & 4 & 0 \\ 0 & 3 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

### Step 3: Reduced Echelon Form

- $R_1 \to \frac{1}{3}R_1$, and $R_2 \to \frac{1}{3}R_2$ :

$$\left(\begin{array}{ccc|c} 1 & \frac{5}{3} & \frac{4}{3} & 0 \\ 0 & 1 & \frac{8}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

### Step 4: Solve for Variables

Let $x_3 = t$, where $t$ is a free parameter:

$$x_2 + \frac{8}{3}x_3 = 0 \implies x_2 = -\frac{8}{3}t$$
$$x_1 + \frac{5}{3}x_2 + \frac{4}{3}x_3 = 0 \implies x_1 = \frac{4}{3}t$$

### Solution Set

The solution set is:

$$\text{Span}\left\{\begin{pmatrix} \frac{4}{3} \\ -\frac{8}{3} \\ 1 \end{pmatrix}\right\}$$

There are infinitely many solutions, indicating the presence of non-trivial solutions.

## Nonhomogeneous Linear System

### Definition

A **nonhomogeneous system** has the form:

$$Ax = b, \quad b \neq 0$$

If there is a particular solution $p$, the general solution of $Ax = b$ is given by:

$$\{p + u : u \text{ is a solution of } Ax = 0\}$$

### Example: Nonhomogeneous System

Consider the system:

$$\begin{aligned} 3x_1 + 5x_2 - 4x_3 &= 7 \\ -3x_1 - 2x_2 + 4x_3 &= -1 \\ 6x_1 + x_2 - 8x_3 &= 4 \end{aligned}$$

### Step 1: Create the Augmented Matrix

$$\left(\begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & 4 \end{array}\right)$$

### Step 2: Row Operations

- $R_2 \to R_2 + R_1:$

$$\left(\begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ 0 & 3 & 0 & 6 \\ 6 & 1 & -8 & 4 \end{array}\right)$$

- $R_3 \to R_3 - 2R_1:$

$$\left(\begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ 0 & 3 & 0 & 6 \\ 0 & -9 & 0 & -10 \end{array}\right)$$

- $R_3 \to R_3 + 3R_2:$

$$\left(\begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ 0 & 3 & 0 & 6 \\ 0 & 0 & 0 & 8 \end{array}\right)$$

### Step 3: Solve for Variables

$x_1 = 3, x_2 = 2, x_3 = 3$ is a particular solution, so:

$$p = \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix}$$

### General Solution

The general solution is:

$$\begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix} + t \begin{pmatrix} \frac{4}{3} \\ -\frac{8}{3} \\ 1 \end{pmatrix}$$

## Linear Independence

### Definition

Let $v_1, \ldots, v_k$ be vectors in $\mathbb{R}^n$. They are **linearly independent** if:

$$c_1 v_1 + \cdots + c_k v_k = 0 \implies c_1 = \cdots = c_k = 0$$

If there exists at least one nonzero $c_i$, the vectors are linearly **dependent**.

### Example: Linear Independence

Consider vectors:

$$u = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \quad v = \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix}, \quad w = \begin{pmatrix} 0 \\ 0 \\ 8 \end{pmatrix}$$

To check for linear independence, set:

$$c_1u + c_2v + c_3w = 0$$

This gives:

$$\begin{aligned} 2c_1 + 0 + 0 &= 0 \\ c_1 + 5c_2 + 0 &= 0 \\ 0 + 0 + 8c_3 &= 0 \end{aligned}$$

The only solution is $c_1 = c_2 = c_3 = 0$, so $u, v, w$ are linearly independent.

## Fact 1

Two vectors $u$ and $v$ in $\mathbb{R}^n$ are **linearly dependent** if and only if one is a multiple of the other.

## Fact 2

The homogeneous equation $Ax = 0$ has a non-trivial solution if and only if the equation has at least one free variable.

## Theorem 5

Suppose the equation $A\mathbf{x} = \mathbf{b}$ is consistent for some given $\mathbf{b}$, and let $\mathbf{p}$ be a solution. Then the solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form

$$\mathbf{w} = \mathbf{p} + \mathbf{v}_h$$

where $\mathbf{v}_h$ is any solution of the homogeneous equation $A\mathbf{x} = 0$.

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# 6. Linear Algebra Concepts

## Linear Equations in Linear Algebra

### Homogeneous Systems

### Definition

A **homogeneous linear system** has the form:

$$Ax = 0$$

where $A$ is a matrix and $x$ is a vector of variables. The vector $x = 0$ is called the **trivial solution**.

### Solutions of Homogeneous Systems

- **Trivial Solution**: The zero vector $x = 0$.
- **Non-Trivial Solutions**: Occur when there are infinitely many solutions, typically due to the matrix $A$ having fewer pivots than the number of variables.

### Solution Set

The solution set can be expressed as:

$$\text{Span}\{v_1, \ldots, v_k\}$$

where $v_1, \ldots, v_k$ are vectors in $\mathbb{R}^n$.

## Example: Homogeneous System

Consider the system:

$$x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

To determine if the vectors are linearly independent, perform row operations on the augmented matrix:

$$\begin{bmatrix} 1 & 4 & 2 & 0 \\ 2 & 5 & 1 & 0 \\ 3 & 6 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

## Solution Interpretation

The system is dependent since there are free variables, indicating the existence of non-trivial solutions.

## Nonhomogeneous Systems

## Definition

A **nonhomogeneous linear system** has the form:

$$Ax = b, \quad b \neq 0$$

where $b$ is a nonzero vector.

## General Solution

If $p$ is a particular solution, the general solution is:

$$\{p + u : u \text{ is a solution of } Ax = 0\}$$

## Example: Nonhomogeneous System

Consider the system:

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$$

In matrix form, we can write this system as:

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$$

## Row Reduction

We set up the augmented matrix and perform row operations to solve for $x_1$, $x_2$, and $x_3$.

$$\begin{bmatrix} 1 & 0 & 1 & | & 2 \\ 1 & 1 & 1 & | & 3 \\ 1 & 1 & 0 & | & 2 \end{bmatrix}$$

**Step 1: Make the First Column Leading 1s**

Subtract the first row from the second and third rows:

$$\begin{bmatrix} 1 & 0 & 1 & | & 2 \\ 0 & 1 & 0 & | & 1 \\ 0 & 1 & -1 & | & 0 \end{bmatrix}$$

**Step 2: Simplify the Third Row**

Subtract the second row from the third row:

$$\begin{bmatrix} 1 & 0 & 1 & | & 2 \\ 0 & 1 & 0 & | & 1 \\ 0 & 0 & -1 & | & -1 \end{bmatrix}$$

**Step 3: Make the Third Column Leading 1**

Multiply the third row by -1:

$$\begin{bmatrix} 1 & 0 & 1 & | & 2 \\ 0 & 1 & 0 & | & 1 \\ 0 & 0 & 1 & | & 1 \end{bmatrix}$$

**Step 4: Back-Substitution**

Now, substitute back to solve for each variable:

1. From the third row: $x_3 = 1$
2. From the second row: $x_2 = 1$
3. From the first row: $x_1 + x_3 = 2 \Rightarrow x_1 = 2 - 1 = 1$

## Particular Solution

A particular solution to the system is:

$$\mathbf{p} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

## General Solution

The general solution to the nonhomogeneous system is given by:

$$\mathbf{x} = \mathbf{p} + \mathbf{u}$$

where $\mathbf{u}$ is any solution to the corresponding **homogeneous system** $Ax = 0$.

For our system:

1. The homogeneous system is:

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

   Solving this system would yield solutions for $\mathbf{u}$.

Therefore, the general solution for the nonhomogeneous system would be the particular solution plus any solutions of the homogeneous system.

To complete the solution, let's find the general solution by solving the homogeneous system $Ax = 0$ and adding it to the particular solution.

### Step 5: Solve the Homogeneous System

We already have the row-reduced form of the matrix:

$$\begin{bmatrix} 1 & 0 & 1 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix}$$

This matrix indicates that the only solution to the homogeneous system $Ax = 0$ is the **zero vector**:

$$\mathbf{u} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

### General Solution for the Nonhomogeneous System

Since the solution to the homogeneous system is only the zero vector, the general solution to the nonhomogeneous system $Ax = b$ is simply the particular solution:

$$\mathbf{x} = \mathbf{p} + \mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

### Final Answer

Thus, the unique solution to the nonhomogeneous system is:

$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

In this example, we found that the nonhomogeneous system has a single unique solution because the homogeneous system has only the trivial solution (no free variables).

## Linear Independence

### Definition

A set of vectors $\{v_1, \ldots, v_p\}$ is **linearly independent** if:

$$c_1v_1 + c_2v_2 + \cdots + c_pv_p = 0 \implies c_1 = c_2 = \cdots = c_p = 0$$

If any coefficient $c_i \neq 0$, the vectors are **linearly dependent**.

### Example: Checking Linear Independence

Consider vectors:

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

To check for linear independence, set:

$$c_1v_1 + c_2v_2 + c_3v_3 = 0$$

After solving, if the only solution is $c_1 = c_2 = c_3 = 0$, then the vectors are linearly independent.

### Theorem: Characterization of Linearly Dependent Sets

An indexed set $S = \{v_1, \ldots, v_p\}$ is **linearly dependent** if at least one vector in $S$ is a linear combination of the others.

# Linear Transformations

## Definition

A **linear transformation** $T : \mathbb{R}^n \to \mathbb{R}^m$ is a mapping that satisfies the following two conditions for all vectors $u, v$ in $\mathbb{R}^n$ and all scalars $c$:

1. **Additivity**: $T(u + v) = T(u) + T(v)$
2. **Homogeneity**: $T(cu) = cT(u)$

These properties ensure that the transformation preserves vector addition and scalar multiplication.

### Example 1: Basic Linear Transformation

Consider a transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by:

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + 3x_2 \\ -x_1 + 4x_2 \end{bmatrix}$$

### Check if $T$ is Linear

1. **Additivity**: For vectors $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$:

$$T(u+v) = T\left(\begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}\right) = \begin{bmatrix} 2(u_1 + v_1) + 3(u_2 + v_2) \\ -(u_1 + v_1) + 4(u_2 + v_2) \end{bmatrix}$$

which simplifies to:

$$T(u) + T(v) = \begin{bmatrix} 2u_1 + 3u_2 \\ -u_1 + 4u_2 \end{bmatrix} + \begin{bmatrix} 2v_1 + 3v_2 \\ -v_1 + 4v_2 \end{bmatrix}$$

Thus, $T(u+v) = T(u) + T(v)$.

1. **Homogeneity**: For a scalar $c$ and vector $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$:

$$T(cu) = T\left(\begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix}\right) = \begin{bmatrix} 2cu_1 + 3cu_2 \\ -cu_1 + 4cu_2 \end{bmatrix}$$

which is equivalent to:

$$cT(u) = c\begin{bmatrix} 2u_1 + 3u_2 \\ -u_1 + 4u_2 \end{bmatrix}$$

Hence, $T$ satisfies both properties and is a linear transformation.

## Matrix Representation of Linear Transformations

## Theorem: Matrix of a Linear Transformation

Every linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ can be represented as a matrix $A$, such that:

$$T(x) = Ax$$

where:

- $A$ is an $m \times n$ matrix,
- $x$ is an $n \times 1$ column vector.

## Constructing the Matrix of a Transformation

To find the matrix $A$ of a linear transformation $T$, apply $T$ to the **standard basis vectors** of $\mathbb{R}^n$. The resulting vectors become the **columns** of the matrix $A$.

## Example 2: Finding the Matrix of a Transformation

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by:

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 - 2x_2 \\ 5x_1 + x_2 \end{bmatrix}$$

## Step 1: Apply T to the Standard Basis Vectors

1. Apply $T$ to $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$:

$$T(e_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

1. Apply $T$ to $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$:

$$T(e_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

## Step 2: Form the Matrix A

$$A = \begin{bmatrix} 3 & -2 \\ 5 & 1 \end{bmatrix}$$

Thus, the matrix of the transformation is $A$.

### Example 3: Geometric Interpretation of a Transformation

Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by:

$$T(x) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x$$

This transformation **rotates vectors by 90 degrees counterclockwise**.

### Applying T to a Vector

If $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, then:

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

This confirms the 90-degree rotation effect.

## Properties of Linear Transformations

### 1. Identity Transformation

The **identity transformation** $I : \mathbb{R}^n \to \mathbb{R}^n$ is defined as:

$$I(x) = x$$

The matrix representation of the identity transformation is the **identity matrix** $I_n$, where:

$$I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

## 2. Zero Transformation

The **zero transformation** $Z : \mathbb{R}^n \to \mathbb{R}^m$ maps every vector to the zero vector:

$$Z(x) = 0$$

The matrix representation is a matrix of all zeros.

## 3. Composition of Linear Transformations

If $T_1 : \mathbb{R}^n \to \mathbb{R}^m$ and $T_2 : \mathbb{R}^m \to \mathbb{R}^p$ are linear transformations, their composition $T_2 \circ T_1 : \mathbb{R}^n \to \mathbb{R}^p$ is also a linear transformation, and its matrix is the product of the matrices:

$$[T_2 \circ T_1](x) = A_2(A_1 x)$$

where $A_1$ and $A_2$ are the matrices of $T_1$ and $T_2$, respectively.

## 4. Invertibility of Linear Transformations

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is **invertible** if there exists another transformation $S : \mathbb{R}^n \to \mathbb{R}^n$ such that:

$$T(S(x)) = S(T(x)) = x$$

### Invertible Matrix Theorem (Key Points)

- $T$ is invertible if and only if its matrix $A$ is invertible.
- $A$ is invertible if it has **full rank** (rank = $n$, where $A$ is an $n \times n$ matrix).

### Example 4: Invertibility

Consider the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by:

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 4x_1 + 3x_2 \\ 2x_1 + x_2 \end{bmatrix}$$

The matrix of $T$ is:

$$A = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$$

Calculate the determinant:

$$\det(A) = 4 \cdot 1 - 3 \cdot 2 = -2 \neq 0$$

Since the determinant is non-zero, $T$ is invertible.

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# 7. Linear Transformations, Row Equivalence, and Elementary Matrices

## Example of a Linear Transformation

Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a transformation defined by

$$T(x, y) = (x, 0)$$

To determine if $T$ is linear, we check two properties:

1. **Additivity**: $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
2. **Scalar Multiplication**: $T(c\mathbf{u}) = cT(\mathbf{u})$

Let $\mathbf{u} = (x_1, y_1)$ and $\mathbf{v} = (x_2, y_2)$ :

1. $T(\mathbf{u} + \mathbf{v}) = T((x_1 + x_2, y_1 + y_2)) = (x_1 + x_2, 0) = (x_1, 0) + (x_2, 0) = T(\mathbf{u}) + T(\mathbf{v})$
2. $T(c\mathbf{u}) = T((cx_1, cy_1)) = (cx_1, 0) = c(x_1, 0) = cT(\mathbf{u})$

Since both properties hold, $T$ is a **linear transformation**.

# Row Equivalence and Elementary Matrices

## Definition of Row Equivalence

Two matrices $A$ and $B$ of size $m \times n$ are **row equivalent** if one can be transformed into the other through a sequence of row operations. This means that $A$ and $B$ are row equivalent **if and only if** their **reduced row echelon forms** are the same.

## Elementary Matrices

An **elementary matrix** $E$ of size $n \times n$ is a matrix that can be obtained from the **identity matrix** $I_n$ by performing a single row operation.

## Examples of Elementary Matrices

1. **Row Swap**: Swapping two rows of $I_3$ :

$$E = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2. **Row Scaling**: Multiplying the second row of $I_3$ by $3$ :

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

3. **Row Addition**: Adding the first row of $I_3$ to the second row:

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

## Property of Elementary Matrices

When an elementary matrix $E$ is multiplied by a matrix $A$ (on the left), the result is equivalent to applying the row operation of $E$ directly to $A$.

# The Matrix Representation of a Linear Transformation

## Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a **linear transformation**. Then there exists a matrix $A$ of size $m \times n$ such that

$$T(\mathbf{x}) = A\mathbf{x} \quad \text{for all } \mathbf{x} \in \mathbb{R}^n.$$

Conversely, if $T : \mathbb{R}^n \to \mathbb{R}^m$ is defined by $T(\mathbf{x}) = A\mathbf{x}$ for some matrix $A$ of size $m \times n$, then $T$ is a linear transformation.

## Proof of the Theorem

Assume that $T : \mathbb{R}^n \to \mathbb{R}^m$ is defined by $T(\mathbf{x}) = A\mathbf{x}$, where $A$ is an $m \times n$ matrix. We show that $T$ is linear by checking the properties of linearity:

1. **Additivity**: For any vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$,

$$T(\mathbf{u}+\mathbf{v}) = A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v} = T(\mathbf{u}) + T(\mathbf{v})$$

1. **Scalar Multiplication**: For any scalar c and vector $\mathbf{u} \in \mathbb{R}^n$,

$$T(c\mathbf{u}) = A(c\mathbf{u}) = c(A\mathbf{u}) = cT(\mathbf{u})$$

Since both properties hold, T is a **linear transformation**.

## Example of a Matrix Representation

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation defined by $T(x, y) = (x + 2y, 3x - y)$.

To find the matrix $A$ such that $T(\mathbf{x}) = A\mathbf{x}$, let:

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$$

Then:

$$T(x, y) = \begin{pmatrix} x + 2y \\ 3x - y \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

So, $A = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}$

# 8. Matrix Operations and Linear Transformations

## Theorem: Linearity and One-to-One Property

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, defined by some matrix $A$ such that $T(x) = Ax$. For example, consider $T : \mathbb{R}^4 \to \mathbb{R}^3$, where $x \to Ax$.

### One-to-One Property

The transformation $T$ is **one-to-one** if and only if $T(x) = 0$ has only the trivial solution $x = 0$.

### Proof: One-to-One Property of Linear Transformations

1. **One Direction**: Assume $T$ is one-to-one. We want to show that $T(x) = 0$ has only the trivial solution.

   Since $T$ is linear, we have $T(0) = 0$. By our assumption that $T$ is one-to-one, this implies that $T(x) = 0$ has only the trivial solution $x = 0$.

2. **Other Direction**: Now, assume $T(x) = 0$ has only the trivial solution $x = 0$. We want to show that $T$ is one-to-one.

   Let $T(u) = T(v)$ for some vectors $u, v \in \mathbb{R}^n$. Then:

$$T(u) - T(v) = 0 \Rightarrow T(u - v) = 0 \quad \text{(since } T \text{ is linear)}$$

Since $T(x) = 0$ has only the trivial solution, $u - v = 0$, which implies $u = v$. Thus, $T$ is one-to-one.

### Onto Property

To be **onto**, the transformation $T$ must map $\mathbb{R}^n$ to cover all of $\mathbb{R}^m$. This means that for every vector $b \in \mathbb{R}^m$, there must exist an $x \in \mathbb{R}^n$ such that $T(x) = b$.

If **not all columns of A contain pivots** (i.e., the columns do not span $\mathbb{R}^m$), then $T$ is **not onto**. This implies that $T(x) = b$ does not have a solution for every $b \in \mathbb{R}^m$.

### Important Fact

For any linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$,

$$T(0) = 0.$$

To see why, note that for any scalar $c \in \mathbb{R}$ and any vector $u \in \mathbb{R}^n$:

$$T(cu) = cT(u).$$

Taking $c = 0$, we get $T(0) = 0$.

## Theorem: Conditions for Onto and One-to-One Properties

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let $A$ be the standard matrix for $T$, meaning $T(x) = Ax$. Then:

1. $T$ is **onto** if and only if the columns of $A$ **span** $\mathbb{R}^m$.
2. $T$ is **one-to-one** if and only if the columns of $A$ are **linearly independent**.

### Proof

1. **Onto Property**: Suppose $T$ is onto. This means that for every $b \in \mathbb{R}^m$, there exists an $x \in \mathbb{R}^n$ such that $Ax = b$. This implies that the columns of $A$ span $\mathbb{R}^m$.

   Conversely, if the columns of $A$ span $\mathbb{R}^m$, then for every $b \in \mathbb{R}^m$, there is a solution $x$ to $Ax = b$. Thus, $T$ is onto.

2. **One-to-One Property**: Suppose $T$ is one-to-one, which means $T(x) = 0$ has only the trivial solution. This implies that the columns of $A$ are linearly independent.

   Conversely, if the columns of $A$ are linearly independent, then $Ax = 0$ has only the trivial solution $x = 0$, implying that $T$ is one-to-one.

## Matrix Operations and Examples

### Definition: Matrix Representation and Indexing

For a matrix $A$ of size $m \times n$, we can represent $A$ by its elements $a_{ij}$, where:

- $1 \leq i \leq m$ (row index)
- $1 \leq j \leq n$ (column index)

### Example: Matrix Definition

Let $A$ and $B$ be $2 \times 2$ matrices defined as follows:

- $A = (a_{ij})$ where $a_{ij} = i + j$
- $B = (b_{ij})$ where $b_{ij} = j^3$

To find $A + B$, calculate each entry:

1. $A = \begin{pmatrix} 1+1 & 1+2 \\ 2+1 & 2+2 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}$
2. $B = \begin{pmatrix} 1^3 & 2^3 \\ 1^3 & 2^3 \end{pmatrix} = \begin{pmatrix} 1 & 8 \\ 1 & 8 \end{pmatrix}$

Adding these matrices:

$$A + B = \begin{pmatrix} 2+1 & 3+8 \\ 3+1 & 4+8 \end{pmatrix} = \begin{pmatrix} 3 & 11 \\ 4 & 12 \end{pmatrix}$$

### Definitions

1. **Onto**: A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **onto** $\mathbb{R}^m$ if for every $\mathbf{b} \in \mathbb{R}^m$, there exists at least one $\mathbf{x} \in \mathbb{R}^n$ such that $T(\mathbf{x}) = \mathbf{b}$.

2. **One-to-One**: A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **one-to-one** if for every $\mathbf{b} \in \mathbb{R}^m$, there is at most one $\mathbf{x} \in \mathbb{R}^n$ such that $T(\mathbf{x}) = \mathbf{b}$.

## Theory

1. **One-to-One Property**: Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then $T$ is **one-to-one** if and only if the equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution ($\mathbf{x} = \mathbf{0}$).

2. **Onto and One-to-One in Terms of the Standard Matrix**:
   Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let $A$ be the standard matrix for $T$. Then:

   - $T$ maps $\mathbb{R}^n$ **onto** $\mathbb{R}^m$ if and only if the columns of $A$ span $\mathbb{R}^m$.
   - $T$ is **one-to-one** if and only if the columns of $A$ are linearly independent.

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# 9. Matrix Properties and Inverses

## Definitions

1. **Symmetric Matrix**: Let $A$ be an $n \times n$ matrix. If $A^T = A$, then $A$ is called a **symmetric matrix**.
2. **Antisymmetric Matrix**: If $A^T = -A$, then $A$ is called an **antisymmetric matrix**.

*Note*: The notation $A^T$ represents the **transpose of matrix** $A$.

### Examples

A symmetric matrix example:

$$A = \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}, \quad A^T = \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix} = A$$

An antisymmetric matrix example:

$$B = \begin{pmatrix} 0 & -5 \\ 5 & 0 \end{pmatrix}, \quad B^T = \begin{pmatrix} 0 & 5 \\ -5 & 0 \end{pmatrix} = -B$$

## Notes and Remarks

### Power of a Matrix

For an $n \times n$ matrix $A$, the $k$-th power of $A$, denoted $A^k$, is defined as the matrix $A$ multiplied by itself $k$ times:

$$A^k = A \cdot A \cdot \cdots \cdot A \quad (k \text{ times})$$

**Example**: Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Then

$$A^2 = A \cdot A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

Similarly, $A^3$ would be calculated as $A \cdot A \cdot A$.

### Matrix Multiplication Non-Commutativity

In general, $AB \neq BA$ for matrices $A$ and $B$. However, there are special cases where $AB = BA$.

**Example (Non-Commutative Case)**:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Then

$$AB = \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix} \neq BA = \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}$$

## Properties of Transpose

For an $n \times n$ matrix $A$ and any scalar $r$, the following properties hold:

1. $(A^T)^T = A$
2. $(A + B)^T = A^T + B^T$
3. $(rA)^T = rA^T$
4. $(AB)^T = B^T \cdot A^T$

### Example Proof of Property 4: $(AB)^T = B^T \cdot A^T$

To prove $(AB)^T = B^T \cdot A^T$, let $C = AB$ and consider any element $c_{ij}$ of $C$ :

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

The transpose of $C$, denoted $C^T$, has elements $c_{ji}$ :

$$c_{ji} = \sum_{k=1}^{n} b_{ki} a_{kj} = (B^T \cdot A^T)_{ji}$$

Therefore, $(AB)^T = B^T \cdot A^T$.

## Inverse of a Matrix

A square matrix $A$ of size $n \times n$ is **invertible** if there exists an $n \times n$ matrix $B$ such that

$$AB = BA = I_n$$

If $A$ is invertible, its inverse is denoted $A^{-1}$.

### Theorem: Inverse of a Matrix

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If the **determinant** of $A$, $\det(A) = ad - bc \neq 0$, then $A^{-1}$ exists and is given by

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

### Example

Given $A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$, calculate $A^{-1}$ :

1. Compute $\det(A) = 2 \cdot 4 - 3 \cdot 1 = 8 - 3 = 5$.
2. Apply the formula:

$$A^{-1} = \frac{1}{5}\begin{pmatrix} 4 & -3 \\ -1 & 2 \end{pmatrix}$$

## Theorem: Conditions for Invertibility

Let $A$ be an $n \times n$ matrix. $A$ is **invertible** if and only if $\det(A) \neq 0$. In such a case, the inverse $A^{-1}$ satisfies

$$A^{-1} \cdot A = I_n$$

## Application: Solving Linear Systems

Given a linear system $Ax = b$, if $A$ is a square and invertible matrix (i.e., $\det(A) \neq 0$), the unique solution is

$$x = A^{-1}b$$

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# 10. Matrix Inverses and Their Characterizations

## The Inverse of a Matrix

### Definition of an Invertible Matrix

A square matrix $A$ of size $n \times n$ is said to be invertible (or nonsingular) if there exists a matrix $B$ of the same size such that:

$$AB = BA = I_n$$

where $I_n$ is the $n \times n$ identity matrix. The matrix $B$ is called the inverse of $A$, denoted as $A^{-1}$.

### Key Properties of Inverses

If $A$ is invertible, its inverse $A^{-1}$ satisfies the following properties:

- The inverse of $A^{-1}$ is $A$ :

$$(A^{-1})^{-1} = A$$

- The inverse of the product of two invertible matrices $A$ and $B$ is:

$$(AB)^{-1} = B^{-1}A^{-1}$$

- For scalar multiplication:

$$(kA)^{-1} = \frac{1}{k}A^{-1}, \quad \text{for } k \neq 0$$

### Finding the Inverse Using Row Reduction

To find $A^{-1}$, augment the matrix $A$ with the identity matrix $I_n$ and row reduce $[A \mid I_n]$ to $[I_n \mid A^{-1}]$.

Example: For $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, augment $A$ with $I_2$ :

$$[A \mid I_2] = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{bmatrix}$$

Perform row operations until:

$$[I_2 \mid A^{-1}] = \begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 1.5 & -0.5 \end{bmatrix}$$

Thus:

$$A^{-1} = \begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix}$$

### Finding the Inverse Using Determinants

For a square matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the inverse can also be computed using the determinant of $A$. The formula is:

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

where:

$$\det(A) = ad - bc$$

If $\det(A) = 0$, the matrix $A$ is not invertible.

**Example:** For $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, compute:

1. $\det(A) = (1)(4) - (2)(3) = -2$
2. Substitute into the formula:

$$A^{-1} = \frac{1}{-2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix}$$

### Invertibility and Elementary Matrices

Elementary matrices are obtained by performing a single row operation on the identity matrix.

Each elementary matrix is invertible, and its inverse corresponds to the reverse row operation.

A matrix $A$ is invertible if and only if it can be written as a product of elementary matrices:

$$A = E_1 E_2 \cdots E_k$$

## Characterizations of Invertible Matrices

### The Invertible Matrix Theorem

For an $n \times n$ matrix $A$, the following statements are equivalent (all true or all false):

- $A$ is an invertible matrix
- $A$ is row equivalent to the $n \times n$ identity matrix
- $A$ has n pivot positions
- The equation $A\mathbf{x} = 0$ has only the trivial solution
- The columns of $A$ form a linearly independent set
- The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one
- The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b} \in \mathbb{R}^n$

- The columns of $A$ span $\mathbb{R}^n$
- The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$
- There exists an $n \times n$ matrix $C$ such that $CA = I$
- There exists an $n \times n$ matrix $D$ such that $AD = I$
- $A^{\mathrm{T}}$ is an invertible matrix

## Applications of the Invertible Matrix Theorem

1. **Solving Systems of Equations:**
   If $A$ is invertible, the solution to $A\mathbf{x} = \mathbf{b}$ is:

$$\mathbf{x} = A^{-1}\mathbf{b}$$

2. **Linear Transformations:**
   A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is invertible if its standard matrix $A$ is invertible.
3. **Matrix Properties:**
   Using equivalences like determinant and rank, the invertibility of a matrix can be determined without explicitly calculating the inverse.

## Determinants and Invertibility

A square matrix $A$ is invertible if $\det(A) \neq 0$.

The determinant provides a numerical measure of whether the rows (or columns) of $A$ are linearly independent.

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# 11. Determinants and Their Properties

## Introduction to Determinants

The determinant is a scalar value associated with a square matrix. It provides information about matrix properties such as invertibility and the geometric scaling factor of the transformation represented by the matrix. Determinants are used in solving systems of linear equations, computing eigenvalues, and finding areas or volumes.

For a $2 \times 2$ matrix:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

the determinant is defined as:

$$\det(A) = ad - bc$$

For larger matrices, the determinant is calculated recursively using cofactor expansion.

## Definition of Determinants

The determinant of an $n \times n$ matrix $A$ can be computed using a cofactor expansion across any row or down any column.

For the $i$-th row, the determinant is:

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

where $C_{ij} = (-1)^{i+j}\det(A_{ij})$ is the cofactor of the element $a_{ij}$, and $A_{ij}$ is the minor matrix obtained by removing the $i$-th row and $j$-th column.

For the $j$-th column, the determinant is:

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

In general, for any row or column, the determinant can be expressed as:

$$\det(A) = \sum_{k=1}^{n} a_{ik}C_{ik}$$

where $i$ is the chosen row (or column) and $C_{ik}$ are the corresponding cofactors.

## Properties of Determinants

Determinants satisfy the following key properties:

- Interchanging two rows or columns changes the sign of the determinant.
- Multiplying a row or column by a scalar multiplies the determinant by the same scalar.
- Adding a multiple of one row or column to another does not change the determinant.
- The determinant of a triangular matrix (upper or lower) is the product of its diagonal entries:

$$\det(A) = \prod_{i=1}^{n} a_{ii}$$

- If a matrix has a row or column of all zeros, its determinant is zero.

- The determinant of the product of two square matrices is the product of their determinants:

$$\det(AB) = \det(A) \cdot \det(B)$$

- The determinant of the transpose of a matrix is equal to the determinant of the original matrix:

$$\det(A^T) = \det(A)$$

## Row Operations and Determinants

Row operations affect the determinant in the following ways:

- Adding a multiple of one row to another row does not change the determinant.
- Interchanging two rows multiplies the determinant by $-1$.
- Multiplying a row by a scalar $k$ multiplies the determinant by $k$.

## Invertibility and Determinants

A square matrix $A$ is invertible if and only if $\det(A) \neq 0$. This property is a quick and efficient way to check for invertibility.

## Multiplicative Property of Determinants

For any two $n \times n$ matrices $A$ and $B$:

$$\det(AB) = \det(A) \cdot \det(B)$$

## Applications of Determinants

### Cramer's Rule

Cramer's Rule uses determinants to solve systems of linear equations $A\mathbf{x} = \mathbf{b}$ :

$$x_i = \frac{\det(A_i)}{\det(A)}$$

where $A_i$ is the matrix obtained by replacing the $i$-th column of $A$ with the vector $\mathbf{b}$.

### Eigenvalues

The determinant helps find eigenvalues of a matrix $A$ through the characteristic equation:

$$\det(A - \lambda I) = 0$$

### Geometric Interpretation

The determinant describes the scaling factor of a linear transformation:

- For $2 \times 2$ matrices, the determinant gives the signed area of a parallelogram.
- For $3 \times 3$ matrices, the determinant gives the signed volume of a parallelepiped. A determinant of zero indicates that the transformation collapses the space into a lower dimension.

## Example: Determinant Calculation

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{bmatrix}$. Compute $\det(A)$ using cofactor expansion along the first row:

$$\det(A) = 1 \cdot \det \begin{bmatrix} 4 & 5 \\ 0 & 6 \end{bmatrix} - 2 \cdot \det \begin{bmatrix} 0 & 5 \\ 1 & 6 \end{bmatrix} + 3 \cdot \det \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix}$$

Compute each minor:

$$\det \begin{bmatrix} 4 & 5 \\ 0 & 6 \end{bmatrix} = (4)(6) - (0)(5) = 24$$

$$\det \begin{bmatrix} 0 & 5 \\ 1 & 6 \end{bmatrix} = (0)(6) - (1)(5) = -5$$

$$\det \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix} = (0)(0) - (1)(4) = -4$$

Substitute back:

$$\det(A) = 1(24) - 2(-5) + 3(-4) = 24 + 10 - 12 = 22$$

The determinant of $A$ is $22.$

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# 12. Linear Transformations, Determinants, and Midterm Preparation

## Linear Transformations

### Definition

A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is **linear** if:

1. $T(u + v) = T(u) + T(v)$ for all $u, v \in \mathbb{R}^n$
2. $T(cu) = cT(u)$ for all $c \in \mathbb{R}$, $u \in \mathbb{R}^n$

### Example

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined as $T(x, y) = (x, y, 0)$ :

1. $T(u + v) = T((x_1 + x_2), (y_1 + y_2)) = (x_1 + x_2, y_1 + y_2, 0) = T(u) + T(v)$
2. $T(cu) = T(cx, cy) = (cx, cy, 0) = cT(u)$

Thus, $T$ is linear.

# Matrix Representations and Row Equivalence

## Matrix Representation

Given $A = [\mathbf{a}_1\,\mathbf{a}_2\,\ldots\,\mathbf{a}_n]$, the transformation $T(x) = Ax$ can be represented using columns of $A$.

## Linear Independence of Columns

- If $B$ has linearly independent columns, then $AB$ also has linearly independent columns.
- Proof (contradiction):
    1. Assume $AB$ is dependent: $ABx = 0$ for some non-zero $x$.
    2. This implies $Ax = 0$, contradicting independence of $B$'s columns.

# Midterm Preparation Problems and Solutions

## Problem 1

**Find if $T(x, y) = (x + y, x - y, 0)$ is linear:**

- Additivity: $T(u + v) = T((x_1 + x_2), (y_1 + y_2)) = (x_1 + x_2 + y_1 + y_2, x_1 + x_2 - (y_1 + y_2), 0) = T(u) + T(v)$
- Scalar Multiplication: $T(cu) = T(cx, cy) = (cx + cy, cx - cy, 0) = cT(u)$

## Problem 2

**Range of $T : \mathbb{R}^3 \to \mathbb{R}^2$, where $T(x, y, z) = (2x + 3y, x - y + z)$ :**

- Range: $\mathrm{span}\{(2, 1), (3, -1), (0, 1)\} = \mathbb{R}^2$ (columns span $\mathbb{R}^2$).

## Problem 3

**Find $\det(A)$ for $A = \begin{bmatrix} -2 & -7 & -9 \\ 2 & 5 & 6 \\ 1 & 3 & 4 \end{bmatrix}$:**

1. Perform row reduction:

- $R_3 \to R_3 - R_1, R_2 \to R_2 - 2R_1$.

2. Resulting upper triangular form gives $\det(A) = (-1)^r \prod a_{ii}$, where $r$ is the number of row swaps.

# 14. Subspaces and Basis in n-Dimensional Real Spaces

## Subspaces in $\mathbb{R}^n$

A subspace $H \subseteq \mathbb{R}^n$ is a subset that satisfies the following properties:

1. **Zero Vector**: The zero vector, $\mathbf{0}$, is in $H$.
2. **Closure under Addition**: If $\mathbf{u}, \mathbf{v} \in H$, then $\mathbf{u} + \mathbf{v} \in H$.
3. **Closure under Scalar Multiplication**: If $\mathbf{u} \in H$ and $c \in \mathbb{R}$, then $c\mathbf{u} \in H$.

### Examples of Subspaces

1. The set of all vectors in $\mathbb{R}^3$ of the form $(x, 0, 0)$ is a subspace because it satisfies all three properties.
2. The set of solutions to a homogeneous system of linear equations forms a subspace.

## Span of Vectors

Given vectors $\mathbf{u}_1, \ldots, \mathbf{u}_k \in \mathbb{R}^n$, their span is the set of all linear combinations of these vectors:

$$\text{Span}\{\mathbf{u}_1, \ldots, \mathbf{u}_k\} = \{c_1\mathbf{u}_1 + \cdots + c_k\mathbf{u}_k \mid c_1, \ldots, c_k \in \mathbb{R}\}.$$

### Key Properties:

- The span is always a subspace of $\mathbb{R}^n$.
- If $S = \text{Span}\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$, then every element in $S$ is a linear combination of the given vectors.

### Proof that Span is a Subspace

Let $S = \text{Span}\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$:

1. **Zero Vector**: Setting $c_1 = c_2 = \cdots = c_k = 0$, we get $\mathbf{0} \in S$.
2. **Closure under Addition**: If $\mathbf{v} = \sum c_i\mathbf{u}_i$ and $\mathbf{w} = \sum d_i\mathbf{u}_i$, then:

$$\mathbf{v} + \mathbf{w} = \sum (c_i + d_i)\mathbf{u}_i \in S.$$

3. **Closure under Scalar Multiplication**: If $\mathbf{v} = \sum c_i\mathbf{u}_i$ and $k \in \mathbb{R}$, then:

$$k\mathbf{v} = \sum (kc_i)\mathbf{u}_i \in S.$$

## Column Space and Null Space

### Column Space ($\text{Col}(A)$)

For a matrix $A = [\mathbf{a}_1, \ldots, \mathbf{a}_n]$, the column space is:

$$\text{Col}(A) = \text{Span}\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}.$$

- $\text{Col}(A) \subseteq \mathbb{R}^m$.
- It represents all linear combinations of the columns of $A$.

### Null Space ($\text{Null}(A)$)

The null space of a matrix $A$ is:

$$\text{Null}(A) = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\}.$$

- $\text{Null}(A) \subseteq \mathbb{R}^n$.
- It represents the solution set to the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

### Proof that Null Space is a Subspace

Let $H = \text{Null}(A)$:

1. **Zero Vector**: $A\mathbf{0} = \mathbf{0}$, so $\mathbf{0} \in H$.
2. **Closure under Addition**: If $A\mathbf{u} = \mathbf{0}$ and $A\mathbf{v} = \mathbf{0}$, then:

$$A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} = \mathbf{0}.$$

3. **Closure under Scalar Multiplication**: If $A\mathbf{u} = \mathbf{0}$ and $c \in \mathbb{R}$, then:

$$A(c\mathbf{u}) = c(A\mathbf{u}) = \mathbf{0}.$$

## Basis for a Subspace

A set $B = \{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is a basis for a subspace $H \subseteq \mathbb{R}^n$ if:

1. $B$ spans $H \Rightarrow \text{Span}\{\mathbf{u}_1, \ldots, \mathbf{u}_k\} = H$.
2. $B$ is linearly independent.

### Example: Basis of $\mathbb{R}^3$

The standard basis for $\mathbb{R}^3$ is:

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

### Properties of Basis

1. Every subspace has a basis.
2. The number of vectors in the basis of a subspace is its dimension.

## Dimension of a Subspace

The dimension of a subspace is the number of vectors in its basis.

## Rank-Nullity Theorem

For a matrix $A_{m \times n}$:

$$\text{Rank}(A) + \text{Nullity}(A) = n$$

where:

- $\text{Rank}(A) = \dim(\text{Col}(A))$
- $\text{Nullity}(A) = \dim(\text{Null}(A))$

The upper limit for $\text{Rank}(A)$ is the minimum number of $m$ and $n$. $\text{Rank}(A)$ can be at most $min(m, n)$.

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# 15. Understanding the Rank-Nullity Theorem

### Rank-Nullity Theorem

For a matrix $A_{m \times n}$

$$\text{Rank}(A) + \text{Nullity}(A) = n$$

where:

- $\text{Rank}(A) = \dim(\text{Col}(A))$
- $\text{Nullity}(A) = \dim(\text{Null}(A))$

The upper limit for $\text{Rank}(A)$ is the minimum number of $m$ and $n$. $\text{Rank}(A)$ can be at most $min(m, n)$.

### Theorem

The pivot columns of a matrix form a basis for $Col(A)$.

# 16. Vector Spaces, Subspaces, and Related Concepts

## Vector Spaces

A **vector space** is a set $V$ of objects called vectors, along with two operations: **vector addition** and **scalar multiplication**. These operations satisfy the following ten axioms for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and scalars $c, d \in \mathbb{R}$.

### Axioms of Vector Spaces

### Addition Axioms

1. **Closure under Addition**:

   If $\mathbf{u}, \mathbf{v} \in V$, then $\mathbf{u} + \mathbf{v} \in V$.

2. **Commutativity of Addition**:

   $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.

3. **Associativity of Addition**:

   $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.

4. **Existence of Zero Vector**:

There exists a zero vector $\mathbf{0} \in V$ such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.

5. **Existence of Additive Inverses**:

   For every $\mathbf{u} \in V$, there exists a vector $-\mathbf{u} \in V$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.

### Scalar Multiplication Axioms

1. **Closure under Scalar Multiplication**:

   If $c \in \mathbb{R}$ and $\mathbf{u} \in V$, then $c\mathbf{u} \in V$.

2. **Distributivity of Scalar Multiplication over Vector Addition**:

   $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.

3. **Distributivity of Scalar Multiplication over Scalar Addition**:

   $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$.

4. **Associativity of Scalar Multiplication**:

   $c(d\mathbf{u}) = (cd)\mathbf{u}$.

5. **Identity Property of Scalar Multiplication**:

   $1\mathbf{u} = \mathbf{u}$.

## Examples of Vector Spaces

### Polynomials of Degree at Most $n$

Let $P_n = \{a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n : a_i \in \mathbb{R}\}$. This set is a vector space.

**Example**:

- $p(t) = 1 + t^2 \in P_3$.
- The zero vector is the zero polynomial $p(t) = 0$.
- Check closure under addition and scalar multiplication:
  - If $p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$ and $q(t) = b_0 + b_1 t + b_2 t^2 + b_3 t^3$, then:

$$p(t) + q(t) = (a_0 + b_0) + (a_1 + b_1)t + (a_2 + b_2)t^2 + (a_3 + b_3)t^3.$$

### Real-Valued Functions

Let $V$ be the set of all real-valued functions defined on a domain $D$. Examples include functions like $\sin(x), \cos(x)$, and $x^2$. This set satisfies all vector space axioms.

### The Set of Sequences

Let $S = \{\dots, y_{-2}, y_{-1}, y_0, y_1, y_2, \dots\}$, where $y_i \in \mathbb{R}$. This is a vector space under pointwise addition and scalar multiplication.

## Subspaces

### Definition

A subset $H \subseteq V$ is a **subspace** of a vector space $V$ if:

1. $\mathbf{0} \in H$,
2. $\mathbf{u} + \mathbf{v} \in H$ for all $\mathbf{u}, \mathbf{v} \in H$,
3. $c\mathbf{u} \in H$ for all $c \in \mathbb{R}$ and $\mathbf{u} \in H$.

### Theorem

If $\mathbf{v}_1, \dots, \mathbf{v}_k \in V$, then $H = \text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is a subspace of $V$.

**Proof**:

1. The zero vector $\mathbf{0} \in H$ since $H$ contains all linear combinations of $\mathbf{v}_1, \dots, \mathbf{v}_k$, and $0 \cdot \mathbf{v}_1 + \cdots + 0 \cdot \mathbf{v}_k = \mathbf{0}$.
2. Closure under addition: If $\mathbf{u}, \mathbf{w} \in H$, then $\mathbf{u} = c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k$ and $\mathbf{w} = d_1\mathbf{v}_1 + \cdots + d_k\mathbf{v}_k$. Then:

$$\mathbf{u} + \mathbf{w} = (c_1 + d_1)\mathbf{v}_1 + \cdots + (c_k + d_k)\mathbf{v}_k \in H.$$

3. Closure under scalar multiplication: If $\mathbf{u} \in H$, then $\mathbf{u} = c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k$. For any scalar $a$,

$$a\mathbf{u} = (ac_1)\mathbf{v}_1 + \cdots + (ac_k)\mathbf{v}_k \in H.$$

**Example**:

Let $H = \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} : a, b \in \mathbb{R} \right\}$. Show $H$ is a subspace of $M_{2\times 2}$.

1. $\mathbf{0} \in H$: $a = 0, b = 0$, so $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in H$.
2. Closure under addition and scalar multiplication follows from the form of the matrices.

## Dimension of a Matrix Space

The set of $m \times n$ matrices $M_{m\times n}$ forms a vector space. Its dimension is $mn$.

**Example**:

For $M_{2\times 3}$, the dimension is $2 \cdot 3 = 6$, and a basis consists of matrices with a single entry as 1 and others as 0.

## Invertible Matrix Theorem (Continued)

Let $A \in \mathbb{R}^{n\times n}$. The following are equivalent:

1. $A$ is invertible,
2. The columns of $A$ form a basis for $\mathbb{R}^n$,
3. $\mathrm{Col}(A) = \mathbb{R}^n$,
4. $\mathrm{rank}(A) = n$,
5. $\dim(\mathrm{Nul}(A)) = 0$,
6. $\mathrm{Nul}(A) = \{\mathbf{0}\}$.

# 17. Subspaces, Rowspace, and Linear Transformations

## Theorem: Span as a Subspace

**Statement**: If $v_1, \ldots, v_k$ are in a vector space $V$, then $\text{Span}\{v_1, \ldots, v_k\}$ is a subspace of $V$.

### Proof

To show $H = \text{Span}\{v_1, \ldots, v_k\}$ is a subspace, verify the three subspace criteria:

1. **Zero Vector in $H$:**

   The zero vector is in $H$ because:

$$0 = 0 \cdot v_1 + 0 \cdot v_2 + \cdots + 0 \cdot v_k \in H.$$

2. **Closed under Addition:**

   Let $u, w \in H$. Then:

$$u = c_1 v_1 + c_2 v_2 + \cdots + c_k v_k, \quad w = d_1 v_1 + d_2 v_2 + \cdots + d_k v_k.$$

   Adding $u$ and $w$:

$$u + w = (c_1 + d_1)v_1 + (c_2 + d_2)v_2 + \cdots + (c_k + d_k)v_k \in H.$$

3. **Closed under Scalar Multiplication:**

   Let $u \in H$ and $c \in \mathbb{R}$. Then:

$$u = c_1 v_1 + c_2 v_2 + \cdots + c_k v_k.$$

Scaling $u$:

$$cu = (cc_1)v_1 + (cc_2)v_2 + \cdots + (cc_k)v_k \in H.$$

Thus, $H = \mathrm{Span}\{v_1, \ldots, v_k\}$ is a subspace of $V$.

## Example 1: Subspaces of $\mathbb{R}^2$

Let $W = \{[x\ y] : xy \geq 0\} \subseteq \mathbb{R}^2$.

**Question**: Is $W$ a subspace of $\mathbb{R}^2$?

### Solution

To check if $W$ is a subspace:

1. $0 \in W$: True, as $x = 0$ and $y = 0$ satisfy $xy = 0 \geq 0$.
2. Closed under addition:
   Let $u = [x_1\ y_1]$, $w = [x_2\ y_2]$ in $W$, so $x_1y_1 \geq 0$ and $x_2y_2 \geq 0$.
   However, $(x_1 + x_2)(y_1 + y_2) \geq 0$ does not always hold (e.g., $[-3, -10] + [5, 5] = [2, -5] \notin W$).
3. Closed under scalar multiplication

   If $u = [x, y] \in W$, then for any scalar $c \in \mathbb{R}$, $cu = [cx, cy] \in W$ because $(cx)(cy) = c^2(xy) \geq 0$.
   Thus, $W$ is closed under scalar multiplication.

Thus, $W$ is **not a subspace** of $\mathbb{R}^2$.

## Example 2: Subspace of $\mathbb{R}^4$

Let $H = \{[4a + 3b, 0, a + b + c, c - 2a] : a, b, c \in \mathbb{R}\} \subseteq \mathbb{R}^4$.

**Question**: Is $H$ a subspace of $\mathbb{R}^4$?

### Solution

1. **Zero Vector in $H$:**

   Set $a = 0, b = 0, c = 0$:

   $$[4(0) + 3(0), 0, 0 + 0 + 0, 0 - 2(0)] = [0, 0, 0, 0] \in H.$$

2. **Closed under Addition:**

   Let $u = [4a_1 + 3b_1, 0, a_1 + b_1 + c_1, c_1 - 2a_1]$ and $w = [4a_2 + 3b_2, 0, a_2 + b_2 + c_2, c_2 - 2a_2]$.
   Adding:

   $$u + w = [4(a_1 + a_2) + 3(b_1 + b_2), 0, (a_1 + a_2) + (b_1 + b_2) + (c_1 + c_2), (c_1 + c_2) - 2(a_1 + a_2)].$$

   Since $a_1 + a_2, b_1 + b_2, c_1 + c_2 \in \mathbb{R}$, $u + w \in H$.

3. **Closed under Scalar Multiplication:**

   Let $u = [4a + 3b, 0, a + b + c, c - 2a]$ and $k \in \mathbb{R}$. Then:

$$ku = [k(4a+3b), 0, k(a+b+c), k(c-2a)].$$

Since $ka, kb, kc \in \mathbb{R}$, $ku \in H$.

Thus, $H$ is a **subspace** of $\mathbb{R}^4$.

## Rowspace of a Matrix

### Definition

The **rowspace** of a matrix $A$ is the subspace of $\mathbb{R}^n$ spanned by the rows of $A$. It is written as:

$$\text{Row}(A) = \text{Span}\{\text{rows of } A\}.$$

### Note

The column space of $A^T$ is the rowspace of $A$:

$$\text{Col}(A^T) = \text{Row}(A).$$

## Basis for the Rowspace

To find a basis for $\text{Row}(A)$, take the non-zero rows of the row-echelon form (or reduced row-echelon form) of $A$.

## Theorem: Column Space

Let $A$ be an $m \times n$ matrix. The column space of $A$ is defined as:

$$\text{Col}(A) = \{b \in \mathbb{R}^m : b = Ax \text{ for some } x \in \mathbb{R}^n\}.$$

### Example

Let $u = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$. Is $u$ in the span of:

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 5 & -8 \\ -1 & 1 & 3 \end{bmatrix}?$$

Solve $Ax = u$, where:

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 5 & -8 \\ -1 & 1 & 3 \end{bmatrix}, \quad u = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}.$$

Write the augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ 2 & 5 & -8 & 2 \\ -1 & 1 & 3 & 5 \end{array}\right].$$

Reduce to row-echelon form and check consistency.

# Linear Transformations

## Definition

Let $V$ and $W$ be vector spaces. A map $T : V \to W$ is **linear** if:

1. $T(a + b) = T(a) + T(b)$ for all $a, b \in V$,
2. $T(ca) = cT(a)$ for all $c \in \mathbb{R}$ and $a \in V$.

## Kernel and Range

- **Kernel**:

  The **kernel** of a linear transformation $T : V \to W$ is the set of all vectors in $V$ that map to the zero vector in $W$:

  $$\ker(T) = \{v \in V : T(v) = 0\}.$$

  For a matrix $A \in \mathbb{R}^{m\times n}$, the kernel (also called the **null space**) is:

  $$\ker(A) = \{x \in \mathbb{R}^n : Ax = 0\}.$$

  The kernel is always a **subspace** of the domain of $T$ or $A$.

  **Proof**:

  - $0 \in \ker(T)$, since $T(0) = 0$.
  - Closed under addition: If $u, v \in \ker(T)$, then $T(u + v) = T(u) + T(v) = 0 + 0 = 0$.
  - Closed under scalar multiplication: If $u \in \ker(T)$ and $c \in \mathbb{R}$, then $T(cu) = cT(u) = c \cdot 0 = 0$.

- **Range**:

  $$\mathrm{Range}(T) = \{T(u) : u \in V\} \subseteq W.$$

  The range is a subspace of $W$.

  **Proof**:

  - $0 \in \mathrm{Range}(T)$, since $T(0) = 0$.
  - Closed under addition and scalar multiplication follow similarly.

# Example: Linear Transformation

Let $T : P_2 \to \mathbb{R}^2$, where:

$$T(p) = \begin{bmatrix} p(0) \\ p(1) \end{bmatrix}.$$

1. **Show $T$ is Linear**:
   - $T(p+q) = \begin{bmatrix} (p+q)(0) \\ (p+q)(1) \end{bmatrix} = \begin{bmatrix} p(0)+q(0) \\ p(1)+q(1) \end{bmatrix} = T(p) + T(q).$
   - $T(cp) = \begin{bmatrix} cp(0) \\ cp(1) \end{bmatrix} = c \begin{bmatrix} p(0) \\ p(1) \end{bmatrix} = cT(p).$
2. **Find $\ker(T)$:**
   Solve $T(p) = 0$, where $p(t) = a + bt + ct^2$.
   $T(p) = \begin{bmatrix} a \\ a+b+c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$
   From $a = 0$ and $a + b + c = 0$, we get $b + c = 0$.
   So $\ker(T) = \{bt + ct^2 : b + c = 0\} = \text{Span}\{t - t^2\}$.
   $$\dim(\ker(T)) = 1.$$
3. **Find $\text{Range}(T)$:**
   The range is:
   $$\text{Range}(T) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}.$$
   $$\dim(\text{Range}(T)) = 2.$$

---

# 18. Basis and Subspaces

## Theorem: Basis of a Subspace

Let $H$ be a subspace of $V$. If $\dim(H) = k$, then $k$ vectors in $H$ form a basis for $H$ if they are:

1. Linearly independent, or
2. They span $H$.

## Problem: Analyzing Subspace $H$

Let:

$$H = \left\{ \begin{bmatrix} a & 2a \\ b & 0 \end{bmatrix} : a, b \in \mathbb{R} \right\} \leq M_{2\times 2}.$$

### 1. Show that $H$ is a Subspace of $M_{2\times 2}$

To prove $H$ is a subspace, verify the three conditions:

1. **Zero Vector**: The zero matrix is in $H$:

$$\text{Let } a = 0, b = 0 \quad \Rightarrow \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in H.$$

2. **Closed under Addition**: Let $A = \begin{bmatrix} a_1 & 2a_1 \\ b_1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} a_2 & 2a_2 \\ b_2 & 0 \end{bmatrix}$ be in $H$. Then:

$$A + B = \begin{bmatrix} a_1 + a_2 & 2(a_1 + a_2) \\ b_1 + b_2 & 0 \end{bmatrix}.$$

   Since $a_1 + a_2 \in \mathbb{R}$ and $b_1 + b_2 \in \mathbb{R}$, $A + B \in H$.

3. **Closed under Scalar Multiplication**: Let $A = \begin{bmatrix} a & 2a \\ b & 0 \end{bmatrix}$ and $c \in \mathbb{R}$. Then:

$$cA = \begin{bmatrix} ca & 2(ca) \\ cb & 0 \end{bmatrix}.$$

   Since $ca \in \mathbb{R}$ and $cb \in \mathbb{R}$, $cA \in H$.

Thus, $H$ is a subspace of $M_{2\times 2}$.

## 2. Write a Basis for $H$

The general form of matrices in $H$ is:

$$\begin{bmatrix} a & 2a \\ b & 0 \end{bmatrix} = a \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

The matrices:

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

are linearly independent and span $H$. Hence, they form a basis for $H$.

## 3. Dimension of $H$

The dimension of $H$ is the number of vectors in the basis:

$$\dim(H) = 2.$$

# 19. Coordinate Mapping, Isomorphism, and Linear Independence

## Coordinate Mapping Theorem

### Theorem

Let $V$ be an $n$-dimensional vector space with basis $\mathcal{B} = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$. *The **coordinate mapping**:*

$$\mathcal{C}\mathcal{B} : V \to \mathbb{R}^n$$

*is defined as:*

$$\mathcal{C}\mathcal{B}(\mathbf{v}) = [\mathbf{v}]\mathcal{B} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

where $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$.

### Properties

1. **Linear Transformation**: The coordinate mapping $\mathcal{C}_{\mathcal{B}}$ is a linear transformation.
2. **1-to-1**: $\mathcal{C}_{\mathcal{B}}$ is one-to-one because distinct vectors in $V$ have distinct coordinate vectors.
3. **Onto**: $\mathcal{C}_{\mathcal{B}}$ is onto because every vector in $\mathbb{R}^n$ corresponds to a unique vector in $V$.

### Isomorphism

The coordinate mapping $\mathcal{C}_{\mathcal{B}} : V \to \mathbb{R}^n$ is an **isomorphism**, meaning:

1. $\mathcal{C}_{\mathcal{B}}$ is **linear**.
2. $\mathcal{C}_{\mathcal{B}}$ is **bijective** (1-to-1 and onto).
3. $\mathcal{C}_{\mathcal{B}}$ preserves vector space structure.

## Example: Isomorphism Between $P_n$ and $\mathbb{R}^{n+1}$

### Theorem

The vector space $P_n$ (polynomials of degree at most $n$) is isomorphic to $\mathbb{R}^{n+1}$.

### Proof

1. **Basis for $P_n$**:
   A standard basis for $P_n$ is $\mathcal{B} = \{1, t, t^2, \ldots, t^n\}$.
2. **Coordinate Mapping**:
   Any $p(t) \in P_n$ can be written as:

   $$p(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_n t^n.$$

   Its coordinate vector in $\mathbb{R}^{n+1}$ is:

   $$[p(t)]_{\mathcal{B}} = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}.$$

3. **Linear Transformation**:
   The mapping $\mathcal{C}_{\mathcal{B}} : P_n \to \mathbb{R}^{n+1}$ is linear because:
   - It respects addition: $[p(t) + q(t)]\mathcal{B} = [p(t)]\mathcal{B} + [q(t)]_{\mathcal{B}}$.
   - It respects scalar multiplication: $[cp(t)]\mathcal{B} = c[p(t)]\mathcal{B}$.
4. **1-to-1 and Onto**:
   - $\mathcal{C}_{\mathcal{B}}$ is 1-to-1 because distinct polynomials have distinct coefficients.
   - $\mathcal{C}_{\mathcal{B}}$ is onto because any vector in $\mathbb{R}^{n+1}$ corresponds to a polynomial in $P_n$.

Thus, $P_n \cong \mathbb{R}^{n+1}$.

## Using Coordinate Vectors to Prove Linear Independence

### Problem

Let $\mathcal{B} = \{1, t, t^2\}$ and $p_1(t) = 1 + t, p_2(t) = t + t^2, p_3(t) = 1 + t^2$. Are $p_1(t), p_2(t), p_3(t)$ linearly independent?

### Solution

1. **Coordinate Vectors**:
   Express each polynomial in terms of $\mathcal{B}$:

$$[p_1]\mathcal{B} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad [p_2]\mathcal{B} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad [p_3]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

2. **Matrix Representation**:
   Form a matrix with these vectors as columns:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

3. **Determine Independence**:
   Compute the determinant:

$$\det(A) = 1(1 \cdot 1 - 0 \cdot 1) - 0(1 \cdot 1 - 0 \cdot 1) + 1(1 \cdot 1 - 1 \cdot 1) = 1 + 0 + 0 = 1.$$

Since $\det(A) \neq 0$, the columns are linearly independent, and hence $p_1, p_2, p_3$ are linearly independent.

# Dimensional Relationships in Subspaces

## Theorem

If $H$ is a subspace of $V$ and $\dim(V) = n$, then:

$$\dim(H) \leq \dim(V).$$

## Example

Let $H \subseteq M_{2\times 2}$ be:

$$H = \left\{ \begin{bmatrix} a & 2a \\ b & 0 \end{bmatrix} : a, b \in \mathbb{R} \right\}.$$

1. **Show $H$ is a Subspace**:
   - Zero matrix is in $H$.
   - Closed under addition and scalar multiplication (verify properties).
2. **Basis for $H$**:
   Write a general matrix:

$$\begin{bmatrix} a & 2a \\ b & 0 \end{bmatrix} = a \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

   Basis:

$$\left\{ \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}.$$

3. **Dimension**:

$$\dim(H) = 2.$$

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# 20. Change of Coordinates, Determinants, and Subspaces

## Theorem: Change of Coordinates Matrix

Let $V$ be a vector space, and let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_m\}$ and $\mathcal{C} = \{\mathbf{c}_1, \ldots, \mathbf{c}n\}$ *be two bases for* $V$*.*
*The* ***change of coordinate matrix*** $P_{\mathcal{B}\to\mathcal{C}}$ is defined as the matrix that relates the coordinates of a vector $\mathbf{x}$ with respect to $\mathcal{B}$ and $\mathcal{C}$.

### Formula:

$$[\mathbf{x}]\mathcal{C} = P_{\mathcal{B}\to\mathcal{C}}[\mathbf{x}]_{\mathcal{B}}$$

Here:

- $[\mathbf{x}]_{\mathcal{B}}$ is the coordinate vector of $\mathbf{x}$ with respect to $\mathcal{B}$,
- $[\mathbf{x}]_{\mathcal{C}}$ is the coordinate vector of $\mathbf{x}$ with respect to $\mathcal{C}$,
- $P_{\mathcal{B}\to\mathcal{C}}$ is the change of coordinate matrix.

## Coordinates with Respect to Another Basis

### Existence of Coordinates

Coordinates with respect to any basis $\mathcal{C}$ always exist because basis vectors span the entire vector space $V$.

Given a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, any vector $\mathbf{v} \in V$ can be expressed uniquely as:

$$\mathbf{v} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_n\mathbf{b}_n$$

## Proof: $P_{\mathcal{B}\to\mathcal{C}}$ is Invertible

To prove that $P_{\mathcal{B}\to\mathcal{C}}$ is invertible:

1. **Basis Vectors**: The basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ consists of linearly independent vectors.
2. **Coordinate Representation**: The change of coordinate matrix $P_{\mathcal{B}\to\mathcal{C}}$ is defined as:

$$P_{\mathcal{B}\to\mathcal{C}} = \begin{bmatrix} [\mathbf{b}1]\mathcal{C} & [\mathbf{b}2]\mathcal{C} & \ldots & [\mathbf{b}n]\mathcal{C} \end{bmatrix}$$

3. **Linear Independence**: Since $\mathbf{b}_1, \ldots, \mathbf{b}_n$ are linearly independent in $V$, their coordinate vectors $[\mathbf{b}1]\mathcal{C}, \ldots, [\mathbf{b}n]\mathcal{C}$ are also linearly independent in $\mathbb{R}^n$.
4. **Invertibility**: A matrix with linearly independent columns is invertible. Hence, $P_{\mathcal{B}\to\mathcal{C}}$ is invertible.

## Determinant Rules for an Invertible Matrix

Let $A$ be an $n \times n$ matrix:

1. **Scaling a Matrix**:
   If $A$ is a square matrix, then for any scalar $c$

$$\det(cA) = c^n \cdot \det(A)$$

   where $n$ is the dimension of $A$.

   **Example**:

   If $A$ is a $5 \times 5$ matrix and $\det(A) = 3$, then:

$$\det(2A) = 2^5 \cdot \det(A) = 32 \cdot 3 = 96$$

2. **Determinant of a Product**:
   If $A$ and $B$ are $n \times n$ matrices:

$$\det(AB) = \det(A) \cdot \det(B)$$

3. **Determinant of an Inverse**:
   If $A$ is invertible:

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

4. **Row Operations**:
   - Swapping two rows multiplies the determinant by $-1$.
   - Multiplying a row by $k$ scales the determinant by $k$.
   - Adding a multiple of one row to another does not change the determinant.

# Subspaces of a Vector Space

## Theorem: Subspace Conditions

Let $V$ be a vector space, and let $H_1$ and $H_2$ be subspaces of $V$:

1. **Union of Subspaces**:

$$H_1 \cup H_2 \text{ may not be a subspace of } V$$

   **Counterexample**:
   Let $H_1 = \text{Span}\{\mathbf{u}\}$ and $H_2 = \text{Span}\{\mathbf{v}\}$, where $\mathbf{u}$ and $\mathbf{v}$ are linearly independent.
   - $\mathbf{u} \in H_1$, $\mathbf{v} \in H_2$,
   - $\mathbf{u} + \mathbf{v} \notin H_1 \cup H_2$, so $H_1 \cup H_2$ is not closed under addition.

2. **Intersection of Subspaces**:

$$H_1 \cap H_2 \text{ is always a subspace of } V$$

   - The intersection $H_1 \cap H_2$ contains the zero vector.

- It is closed under addition and scalar multiplication.

## Summary

1. **Change of Coordinates**:
   - The change of coordinate matrix $P_{\mathcal{B}\to\mathcal{C}}$ relates the coordinates of a vector $\mathbf{x}$ in two bases:

   $$[\mathbf{x}]\mathcal{C} = P_{\mathcal{B}\to\mathcal{C}}[\mathbf{x}]_{\mathcal{B}}$$

   - $P_{\mathcal{B}\to\mathcal{C}}$ is invertible.
2. **Determinant Rules**:
   - $\det(cA) = c^n \det(A)$,
   - $\det(AB) = \det(A) \cdot \det(B)$,
   - Row operations affect the determinant as described.
3. **Subspaces**:
   - $H_1 \cup H_2$ is not always a subspace,
   - $H_1 \cap H_2$ is always a subspace.

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# 21. Solutions to Linear Algebra Problems: Preparation for MT 2

📌 **Important Note:**

"I couldn't attend this lecture where students had an open Q&A session with the professor. To ensure I stay on track, I've included the solutions to some of the problems in Fall 2023 Midterm 2 here as a reference."

## Problem 1

Let P_2 denote the vector space of all polynomials of degree at most two. The sets

$$\mathcal{B} = \{1+2t, -2t+2t^2, 3+t+t^2\} \quad \text{and} \quad \mathcal{C} = \{1, t, t^2\}$$

are two bases for $P_2$.

### (a) Find the $\mathcal{C}$-coordinate vector of each of the polynomials in the basis $\mathcal{B}$.

**Solution:**

To find the $\mathcal{C}$-coordinate vectors of the polynomials in $\mathcal{B}$, express each polynomial as a linear combination of the basis $\{1, t, t^2\}$. The coefficients of the linear combinations will form the $\mathcal{C}$-coordinate vectors.

1. **For $1 + 2t$:**

   Express $1 + 2t$ as:

   $$1 + 2t = 1 \cdot 1 + 2 \cdot t + 0 \cdot t^2.$$

   The coefficients are $[1, 2, 0]_{\mathcal{C}}$.

2. **For $-2t + 2t^2$:**

   Express $-2t + 2t^2$ as:

   $$-2t + 2t^2 = 0 \cdot 1 - 2 \cdot t + 2 \cdot t^2.$$

   The coefficients are $[0, -2, 2]_{\mathcal{C}}$.

3. **For $3 + t + t^2$:**

   Express $3 + t + t^2$ as:

   $$3 + t + t^2 = 3 \cdot 1 + 1 \cdot t + 1 \cdot t^2.$$

   The coefficients are $[3, 1, 1]_{\mathcal{C}}$.

The $\mathcal{C}$-coordinate vectors of the polynomials in $\mathcal{B}$ are:

$$[1, 2, 0], \ [0, -2, 2], \ [3, 1, 1].$$

## (b) Find the change-of-coordinates matrix from the basis $\mathcal{B}$ to the basis $\mathcal{C}$.

**Solution:**

The change-of-coordinates matrix $P_{\mathcal{B} \to \mathcal{C}}$ is formed by placing the $\mathcal{C}$-coordinate vectors of the polynomials in $\mathcal{B}$ as the columns of a matrix.

The $\mathcal{C}$-coordinate vectors from part (a) are:

$$\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ -2 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}.$$

Thus, the change-of-coordinates matrix is:

$$P_{\mathcal{B}\to\mathcal{C}} = \begin{bmatrix} 1 & 0 & 3 \\ 2 & -2 & 1 \\ 0 & 2 & 1 \end{bmatrix}.$$

# Problem 3

Let

$$A = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 1 & 3 \\ 0 & -1 & 3 \end{bmatrix}.$$

## (a) Find the determinant of $A$ by expanding along the third column of $A$.

**Solution:**

To compute $\det(A)$, expand along the third column:

$$\det(A) = (-2) \cdot \det \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} - 3 \cdot \det \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} + 3 \cdot \det \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

1. Compute each minor determinant:

   - First minor:

$$\det \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} = (-1)(-1) - (1)(0) = 1.$$

   - Second minor:

$$\det \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} = (1)(-1) - (1)(0) = -1.$$

   - Third minor:

$$\det \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = (1)(1) - (1)(-1) = 2.$$

2. Substitute into the determinant formula:

$$\det(A) = (-2)(1) - 3(-1) + 3(2).$$

3. Simplify:

$$\det(A) = -2 + 3 + 6 = 7.$$

**Final Answer:**

$$\det(A) = 7.$$

## Problem 4

Let $M_{2\times 2}$ be the vector space of all $2 \times 2$ matrices, and define the linear transformation $T : \mathbb{R}^2 \to M_{2\times 2}$ by:

$$T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} 0 & a - 3b \\ a - 3b & 0 \end{bmatrix}.$$

Let

$$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

### (a) Calculate $T(2\mathbf{u} + \mathbf{v})$.

**Solution:**

1. Compute $2\mathbf{u} + \mathbf{v}$:

$$2\mathbf{u} + \mathbf{v} = 2\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}.$$

2. Apply $T$:

$$T\left(\begin{bmatrix} 3 \\ 7 \end{bmatrix}\right) = \begin{bmatrix} 0 & 3 - 3(7) \\ 3 - 3(7) & 0 \end{bmatrix} = \begin{bmatrix} 0 & -18 \\ -18 & 0 \end{bmatrix}.$$

**Final Answer:**

$$T(2\mathbf{u}+\mathbf{v}) = \begin{bmatrix} 0 & -18 \\ -18 & 0 \end{bmatrix}.$$

### (c) If possible, find a vector $\mathbf{w} \in \mathbb{R}^2$ such that $T(\mathbf{w}) = \begin{bmatrix} 0 & 107 \\ 107 & 0 \end{bmatrix}$.

**Solution:**

Set:

$$T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} 0 & a-3b \\ a-3b & 0 \end{bmatrix} = \begin{bmatrix} 0 & 107 \\ 107 & 0 \end{bmatrix}.$$

From the top-right entry:

$$a - 3b = 107.$$

Solve for $\mathbf{w} = \begin{bmatrix} a \\ b \end{bmatrix}$:

$$a = 107 + 3b.$$

Let $b = 0$:

$$\mathbf{w} = \begin{bmatrix} 107 \\ 0 \end{bmatrix}.$$

**Final Answer:**

$$\mathbf{w} = \begin{bmatrix} 107 \\ 0 \end{bmatrix}.$$

## Problem 5

Mark each statement as True or False by writing $T$ or $F$ inside the box to the left of each statement.

No explanation is needed in this question. Assume that all matrices below are square.

### (a) If $\mathbf{v}$ is an eigenvector with eigenvalue $2$, then $2\mathbf{v}$ is an eigenvector with eigenvalue $4$.

**Answer:** False

Scaling an eigenvector does not change its eigenvalue. The eigenvalue remains associated with the eigenvector regardless of scalar multiplication.

### (b) If two matrices of the same size have the same set of eigenvalues, then they are similar.

**Answer:** False

Having the same eigenvalues does not guarantee similarity. Matrices must also have the same eigenvectors (or equivalent diagonalization properties) to be similar.

### (c) Row operations preserve the linear dependence relations among the rows of a matrix.

**Answer:** False

Row operations may alter linear dependence relations. For example, scaling or replacing rows can introduce or remove dependencies.

### (d) If a set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ spans a finite-dimensional vector space $V$ and $T$ is a set of more than $p$ vectors in $V$, then $T$ is linearly dependent.

**Answer:** True

In a vector space of dimension $p$, any set with more than $p$ vectors must be linearly dependent due to the dimensionality constraint.

### (e) If $0$ is an eigenvalue of a matrix $A$, then $A$ is invertible.

**Answer:** False

If $0$ is an eigenvalue, the determinant of $A$ is $0$, making $A$ singular (not invertible).

### (f) If $A^3 = 0$ for a matrix $A$, then $\det(A) = 0$.

**Answer:** True

If $A^3 = 0$, $A$ is a nilpotent matrix, meaning $A$ is singular. Singular matrices have a determinant of $0$.

### (g) If $\lambda$ is an eigenvalue of an invertible matrix $A$, then $\frac{1}{\lambda}$ is an eigenvalue of $A^{-1}$.

**Answer:** True

For an invertible matrix $A$, if $\mathbf{v}$ is an eigenvector corresponding to $\lambda$, then $A^{-1}\mathbf{v} = \frac{1}{\lambda}\mathbf{v}$.

### (h) For any matrix $A$, we have $\det(A^T) = \frac{1}{\det(A)}$.

**Answer:** False

The determinant of the transpose equals the determinant of the matrix, i.e., $\det(A^T) = \det(A)$. The given statement is incorrect.

### (i) If a matrix $A$ is invertible, then $A$ is diagonalizable.

**Answer:** False

Not all invertible matrices are diagonalizable. Diagonalizability requires that the matrix has enough linearly independent eigenvectors.

### (j) If a matrix $A$ is similar to a matrix $B$, then $A^2$ is similar to $B^2$.

**Answer:** True

If $A$ is similar to $B$, then $A^k$ is similar to $B^k$ for any positive integer $k$. This follows from the similarity transformation property.

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# 22. Eigenvalues, Eigenspaces, and Eigenvectors

## Definitions

### Eigenvalues and Eigenvectors

Let $A$ be an $n \times n$ matrix. A scalar $\lambda \in \mathbb{R}$ is called an **eigenvalue** of $A$ if there exists a non-zero vector $\mathbf{x} \in \mathbb{R}^n$ such that:

$$A\mathbf{x} = \lambda\mathbf{x}.$$

Here:

- $\mathbf{x}$ is called an **eigenvector** corresponding to the eigenvalue $\lambda$.
- The set of all solutions $\mathbf{x} \neq 0$ to the equation $A\mathbf{x} = \lambda\mathbf{x}$ forms the **eigenspace** corresponding to $\lambda$.

### Characteristic Equation

The eigenvalue equation $A\mathbf{x} = \lambda\mathbf{x}$ can be rewritten as:

$$(A - \lambda I)\mathbf{x} = 0,$$

where $I$ is the identity matrix. For non-trivial solutions ($\mathbf{x} \neq 0$), the determinant of $A - \lambda I$ must be zero:

$$\det(A - \lambda I) = 0.$$

This is the **characteristic equation** of $A$.

The **characteristic polynomial** of $A$ is defined as:

$$p(\lambda) = \det(A - \lambda I).$$

Geometric and Algebraic Multiplicities

1. **Algebraic Multiplicity**:
   - The algebraic multiplicity of an eigenvalue $\lambda$ is the number of times $\lambda$ appears as a root of the characteristic polynomial $p(\lambda)$.
2. **Geometric Multiplicity**:
   - The geometric multiplicity of an eigenvalue $\lambda$ is the dimension of the eigenspace corresponding to $\lambda$. This is the number of linearly independent eigenvectors associated with $\lambda$.

**Fact**:
For any eigenvalue $\lambda$ of an $n \times n$ matrix $A$:

$$\text{Geometric Multiplicity of } \lambda \leq \text{Algebraic Multiplicity of } \lambda.$$

Example: Finding Eigenvalues and Eigenspaces

Let:

$$A = \begin{bmatrix} 1 & 2 & 2 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}.$$

**Step 1: Find the Characteristic Polynomial**
Solve $\det(A - \lambda I) = 0$:

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 2 & 2 \\ -3 & -5-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{bmatrix}.$$

Compute $\det(A - \lambda I)$:

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 & 2 \\ -3 & -5-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{vmatrix}.$$

Expand the determinant:

$$\det(A - \lambda I) = (1-\lambda)\left((\lambda+5)(1-\lambda) - 9\right) - 2(-3(1-\lambda) + 9) + 2(-9 - 3(\lambda+5)).$$

Simplify to find the characteristic polynomial $p(\lambda)$.

**Step 2: Solve** $p(\lambda) = 0$
Find the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ (roots of $p(\lambda)$).

**Step 3: Find the Eigenspaces**
For each eigenvalue $\lambda$, solve $(A - \lambda I)\mathbf{x} = 0$ to find the eigenvectors and eigenspaces.

### Theorem: Linear Independence of Eigenvectors

If $v_1, \ldots, v_r$ are eigenvectors corresponding to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ of $A$, then $v_1, \ldots, v_r$ are linearly independent.

## Similarity and Diagonalization

### Similarity

Two $n \times n$ matrices $A$ and $B$ are **similar** if there exists an invertible matrix $P$ such that:

$$P^{-1}AP = B.$$

### Diagonalization

A matrix $A$ is **diagonalizable** if there exists an invertible matrix $P$ such that:

$$P^{-1}AP = D,$$

where $D$ is a diagonal matrix. In this case, $A$ is similar to $D$.

**Theorem**:
An $n \times n$ matrix $A$ is diagonalizable if and only if $A$ has $n$ linearly independent eigenvectors.

## Summary of Key Points

1. Eigenvalues are roots of the characteristic polynomial $p(\lambda) = \det(A - \lambda I)$.
2. Eigenvectors are non-zero solutions to $(A - \lambda I)\mathbf{x} = 0$.
3. The geometric multiplicity of $\lambda$ is the dimension of the eigenspace corresponding to $\lambda$.
4. Geometric multiplicity $\leq$ Algebraic multiplicity.
5. A matrix is diagonalizable if it has $n$ linearly independent eigenvectors.

# 23. Foundations of Invertibility, Similarity, and Diagonalization

These notes cover several key theorems related to:

1. **Invertibility of an $n \times n$ matrix and the eigenvalue 0**.
2. **Similarity of matrices** and its properties.
3. **Diagonalizability** and the role of geometric multiplicities.

## Invertibility and the Eigenvalue 0

### Theorem

> Let $A$ be an $n \times n$ matrix. Then $A$ is invertible if and only if $0$ is not an eigenvalue of $A$.

### Reasoning/Proof Sketch

- By definition, $0$ is an eigenvalue of $A$ if there exists a nonzero vector $\mathbf{v}$ such that

$$A\mathbf{v} = 0 \cdot \mathbf{v} = \mathbf{0}.$$

This means $\mathbf{v}$ is in the null space of $A$, so $A$ is **not** injective (not one-to-one).

- A matrix $A$ is invertible precisely when its determinant is **nonzero**.
- It is a standard fact that the determinant of a matrix equals the product of its eigenvalues (counted with algebraic multiplicities).
  Therefore,
  $$\det(A) = \lambda_1 \cdot \lambda_2 \cdot \ldots \cdot \lambda_n.$$
  If $0$ is one of the $\lambda_i$, then the product is $0$, so $\det(A) = 0$ and $A$ is **not** invertible.
- Conversely, if $A$ is **not** invertible, then $\det(A) = 0$, which means at least one eigenvalue must be $0$.

Hence, the statement is proven.

## Similar Matrices

### Definition

Two $n \times n$ matrices $A$ and $B$ are said to be **similar** if there exists an **invertible** matrix $P$ such that

$$B = P^{-1} A \, P$$

In other words, $A$ and $B$ represent the same linear transformation but in different bases.

## Diagonal Matrices

### Definition

An $n \times n$ matrix $D$ is called **diagonal** if all its off-diagonal entries are $0$, i.e.,

$$D = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{pmatrix}.$$

### When does a matrix have a diagonal form?

A matrix $A$ is said to be **diagonalizable** if there exists an invertible matrix $P$ such that $P^{-1}AP$ is a diagonal matrix. This is intimately connected with the notion of having a full set of **linearly independent eigenvectors**.

$$A = PDP^{-1}, \quad D = P^{-1}AP$$

## Similar Matrices Have the Same Characteristic Polynomial

### Theorem

> If two $n \times n$ matrices $A$ and $B$ are similar, then they have the same characteristic polynomial and consequently the same eigenvalues (with the same algebraic multiplicities).

### Proof Sketch

1. **Similarity Assumption**: Suppose $B = P^{-1}AP$ for some invertible $P$.
2. **Characteristic Polynomial**: The characteristic polynomial of a matrix $M$ is given by

$$p_M(\lambda) = \det(\lambda I - M).$$

3. **Compute $p_B(\lambda)$**:

$$p_B(\lambda) = \det(\lambda I - B) = \det\big(\lambda I - P^{-1}AP\big).$$

4. **Factor Out $P^{-1}$ and $P$**:

$$\lambda I - P^{-1}AP = P^{-1}(\lambda PI - A)P = P^{-1}(\lambda I - A)P$$

   since $PI = P$. Therefore,

$$p_B(\lambda) = \det\big(P^{-1}(\lambda I - A)P\big).$$

5. **Use Multiplicative Property of Determinants**:

$$\det\big(P^{-1}(\lambda I - A)P\big) = \det(P^{-1})\ \det(\lambda I - A)\ \det(P).$$

6. **Invertible** $P$: $\det(P^{-1})\det(P) = 1$, hence

$$p_B(\lambda) = \det(\lambda I - A) = p_A(\lambda).$$

7. **Conclusion**: Since $p_B(\lambda) = p_A(\lambda)$, the eigenvalues (the roots of these polynomials) coincide, including their algebraic multiplicities.

### Converse Is Not True

Having the same set of eigenvalues **does not** necessarily imply that two matrices are similar. They **must** also have the same geometric structure of eigenspaces, Jordan canonical forms, etc.

### Example / Idea for Proof

1. Consider the matrices:

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}.$$

2. Both have the same eigenvalue $\lambda = 2$ (with algebraic multiplicity 2).

3. **Eigenvectors**:
   - $A$ is already diagonal, so it has 2 linearly independent eigenvectors.
   - $B$ is a Jordan block (upper triangular with identical diagonal entries 2). It has **only one** linearly independent eigenvector.

4. Thus, $A$ and $B$ are **not** similar, even though they have the same eigenvalue with the same algebraic multiplicities. The difference lies in their **geometric multiplicities** (the dimensions of the eigenspaces).

## Diagonalizability Criterion

### Theorem

> An $n \times n$ matrix $A$ is diagonalizable over $\mathbb{R}$ (or $\mathbb{C}$) if and only if for each eigenvalue $\lambda_i$, the geometric multiplicity (dimension of the eigenspace corresponding to $\lambda_i$) equals its algebraic multiplicity.

### Proof Sketch

- The **algebraic multiplicity** $m_i$ of an eigenvalue $\lambda_i$ is how many times $\lambda_i$ appears as a root of the characteristic polynomial.
- The **geometric multiplicity** $g_i$ is $\dim(\ker(A - \lambda_i I))$.
- To be able to diagonalize $A$, one must be able to find $n$ linearly independent eigenvectors. Equivalently, one must have a basis consisting entirely of eigenvectors.
- For each eigenvalue $\lambda_i$, you can select $g_i$ linearly independent eigenvectors. The sum over all eigenvalues of $g_i$ must be $n$.
- But $\sum_i g_i \leq \sum_i m_i = n$. Thus, for having a full set (exactly $n$) of linearly independent eigenvectors, you need $g_i = m_i$ for every eigenvalue $\lambda_i$.

## Example

Let $A$ be a $6 \times 6$ matrix with characteristic polynomial

$$p(\lambda) = (\lambda - 3)^2(\lambda + 1)(\lambda - 2)^3.$$

- The eigenvalues are $\lambda = 3$ (with algebraic multiplicity 2), $\lambda = -1$ (with algebraic multiplicity 1), and $\lambda = 2$ (with algebraic multiplicity 3).
- To determine diagonalizability, one must check each eigenvalue's geometric multiplicity:
  - For $\lambda = 3$, we need to see if the dimension of $\ker(A - 3I)$ is 2.
  - For $\lambda = -1$, the dimension of $\ker(A + I)$ must be 1.
  - For $\lambda = 2$, the dimension of $\ker(A - 2I)$ must be 3.
- If, and only if, **all** these dimensions match their respective algebraic multiplicities, $A$ is diagonalizable. Otherwise, it is not.

### Summary Points

1. A matrix is invertible $\iff$ $0$ is not among its eigenvalues.
2. Two matrices are similar if one can be obtained from the other by a similarity transformation $B = P^{-1}AP$.

3. Similar matrices have the same characteristic polynomials, hence the same eigenvalues.
4. The converse is **not** true: having the same eigenvalues (even with same multiplicities) does not imply similarity—one must also compare eigenspace dimensions (geometric multiplicities).
5. A matrix $A$ is diagonalizable if and only if for every eigenvalue $\lambda$, its geometric multiplicity equals its algebraic multiplicity.

# 24. Eigenvalues, Eigenvectors, and Diagonalization

## Diagonalization of Matrices

### Problem Statement

We are tasked to:

1. Diagonalize the matrix $A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$, if possible.
2. Confirm whether $A$ is diagonalizable by verifying if the **geometric multiplicity** equals the **algebraic multiplicity** for each eigenvalue.
3. Find the eigenvalues, eigenvectors, eigenspaces, and matrices $P$ (diagonalizing matrix) and $D$ (diagonal matrix).
4. Compute $A^{51}$.
5. Find $P^{-1}$.
6. Discuss diagonalizability based on algebraic multiplicity and eigenspaces for similar matrices.

Additionally:

- Demonstrate why $A^{-1}$ is diagonalizable if $A$ is diagonalizable and invertible.

# Diagonalization of $A$

## Step 1: Find Eigenvalues

To find eigenvalues:

1. Solve $\det(A - \lambda I) = 0$, where $I$ is the identity matrix.
2. Compute $\det(A - \lambda I)$:

$$\det\left(\begin{bmatrix} 1-\lambda & 3 & 3 \\ -3 & -5-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{bmatrix}\right) = 0.$$

3. Expand and simplify to obtain the **characteristic polynomial**:

$$\det(A - \lambda I) = (\lambda + 2)^2(\lambda - 4).$$

Thus, the eigenvalues are:

- $\lambda_1 = -2$ (algebraic multiplicity 2),
- $\lambda_2 = 4$ (algebraic multiplicity 1).

## Step 2: Find Eigenspaces and Eigenvectors

For each eigenvalue $\lambda$, solve $(A - \lambda I)\mathbf{x} = 0$.

### For $\lambda = -2$:

$$(A - (-2)I) = \begin{bmatrix} 3 & 3 & 3 \\ -3 & -3 & -3 \\ 3 & 3 & 3 \end{bmatrix}.$$

Row reduce:

$$\begin{bmatrix} 3 & 3 & 3 \\ -3 & -3 & -3 \\ 3 & 3 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The eigenspace is spanned by $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

#### For $\lambda = 4$:

$$(A - 4I) = \begin{bmatrix} -3 & 3 & 3 \\ -3 & -9 & -3 \\ 3 & 3 & -3 \end{bmatrix}.$$

Row reduce:

$$\begin{bmatrix} -3 & 3 & 3 \\ -3 & -9 & -3 \\ 3 & 3 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The eigenspace is spanned by $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

### Step 3: Verify Diagonalizability

The algebraic multiplicities of the eigenvalues add up to $n = 3$, and the dimensions of the eigenspaces match their algebraic multiplicities:

- For $\lambda = -2$: Algebraic multiplicity = 2, Geometric multiplicity = 2.
- For $\lambda = 4$: Algebraic multiplicity = 1, Geometric multiplicity = 1.

Since geometric multiplicity equals algebraic multiplicity for all eigenvalues, $A$ is diagonalizable.

### Step 4: Construct $P$ and $D$

1. **Matrix $P$**: Columns are the eigenvectors:

$$P = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}.$$

2. **Matrix $D$**: Diagonal matrix of eigenvalues:

$$D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

## Step 5: Compute $A^{51}$

Using diagonalization:

$$A^{51} = PD^{51}P^{-1}.$$

1. Compute $D^{51}$:

$$D^{51} = \begin{bmatrix} (-2)^{51} & 0 & 0 \\ 0 & (-2)^{51} & 0 \\ 0 & 0 & 4^{51} \end{bmatrix}.$$

2. Compute $A^{51}$ by substituting $P, D^{51}, P^{-1}$.

## Step 6: Find $P^{-1}$

Using the formula for the inverse of a matrix:

$$P^{-1} = \text{adj}(P)/\det(P).$$

1. Compute $\det(P)$:

$$\det(P) = 1(-1-1) - 1(-1-0) + 1(1-0) = -2 + 1 + 1 = 0.$$

2. Compute $P^{-1}$ using standard cofactor and adjoint methods.

## Additional Topics

## Diagonalizability of $A^{-1}$

If $A$ is diagonalizable, $A = PDP^{-1}$. Then:

$$A^{-1} = (PDP^{-1})^{-1} = PD^{-1}P^{-1}.$$

Since $D^{-1}$ is diagonal, $A^{-1}$ is also diagonalizable.

## Formulas for Inverses and Matrix Multiplication

1. **Inverse**: For a $2 \times 2$ matrix:

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

2. **Matrix Multiplication**:
   - For $AB \neq BA$, matrices are not commutative.
   - If $A$ and $B$ are inverses: $AB = I$, $BA = I$.

3. **Inverse of a Transpose**:

   If $A$ is invertible, then so is $A^T$, and the inverse of $A^T$ is the transpose of $A^{-1}$:

$$(A^T)^{-1} = (A^{-1})^T.$$

4. **Inverse of a Product**:

   If $A$ and $B$ are invertible $n \times n$ matrices, then the inverse of $AB$ is the product of the inverses of $A$ and $B$ in reverse order:

$$(AB)^{-1} = B^{-1}A^{-1}.$$

5. **Inverse of an Inverse**:

   If $A$ is invertible, then the inverse of $A^{-1}$ is $A$ itself:

$$(A^{-1})^{-1} = A.$$

6. **Inverse of a Scalar Multiple**:

   For any invertible matrix $A$ and scalar $k \neq 0$, the inverse of $kA$ is given by:

$$(kA)^{-1} = \frac{1}{k} A^{-1}.$$

# 25. Complex Eigenvalues and Diagonalization

## What This Note Is About

We study **complex eigenvalues**, how they arise from the characteristic polynomial, and how to find eigenvectors and diagonalize a matrix (or a linear transformation) when working over the complex field.

## Complex Eigenvalues

> **A complex eigenvalue $\lambda$ of a square matrix $A \in M_{n\times n}(\mathbb{R})$ (or $\mathbb{C}$) is a (possibly non-real) complex number for which there exists a nonzero vector $x$ such that**
>
> $$Ax = \lambda x.$$
>
> **The vector $x$ is called an eigenvector corresponding to $\lambda$.**

### Characteristic Polynomial

> **The characteristic polynomial of an $n \times n$ matrix $A$ is defined by**
>
> $$\det(A - \lambda I).$$

> **Its roots (counted with multiplicities) are the eigenvalues of $A$. Over $\mathbb{C}$, every polynomial splits completely, so an $n \times n$ real matrix always has $n$ complex eigenvalues in total (some could be repeated).**

### Example: Complex Eigenvalues of a $2 \times 2$ Matrix

Suppose

$$A = \begin{pmatrix} 2 & -6 \\ 3 & 4 \end{pmatrix}.$$

1. **Characteristic Polynomial**

$$\det \begin{pmatrix} 2-\lambda & -6 \\ 3 & 4-\lambda \end{pmatrix} = (2-\lambda)(4-\lambda) - (-6)\cdot 3 = \lambda^2 - 6\lambda + 26$$

2. **Eigenvalues**

   Solve $\lambda^2 - 6\lambda + 26 = 0$:

$$\lambda = \frac{6 \pm \sqrt{36 - 4\cdot 26}}{2} = \frac{6 \pm \sqrt{-68}}{2} = 3 \pm \sqrt{17}\, i.$$

   These are complex conjugates.

3. **Eigenvectors**

   For each $\lambda$, solve $(A - \lambda I)x = 0$. Because the eigenvalues are complex, the corresponding eigenvectors will also have complex entries. The space of all such eigenvectors is the **eigenspace** for that $\lambda$.

## Eigenvalues in a Linear Transformation Context

> **Let $V$ be a vector space over $\mathbb{R}$ or $\mathbb{C}$, and let $T : V \to V$ be a linear map. A nonzero vector $x \in V$ is called an eigenvector of $T$ if**
>
> $$T(x) = \lambda x$$
>
> **for some scalar $\lambda$. The number $\lambda$ is an eigenvalue of $T$.**

### Example: A Linear Operator on $P_3$

Let $T : P_3 \to P_3$ be defined by

$$T\big(p(t)\big) = p(0) + p(2)\,t - p(0)\,t^2 - p(2)\,t^3.$$

1. **Case (a)**: $p(t) = 1 - t^2$
   - Compute $p(0) = 1$, $p(2) = 1 - 4 = -3$.
   - Then

     $$T(1 - t^2) = p(0) + p(2)\,t - p(0)\,t^2 - p(2)\,t^3 = 1 - 3t - t^2 + 3t^3.$$

     This is **not** a scalar multiple of $1 - t^2$, so $1 - t^2$ is **not** an eigenvector.
2. **Case (b)**: $p(t) = t - t^3$
   - Compute $p(0) = 0$, $p(2) = 2 - 8 = -6$.
   - Then

     $$T(t - t^3) = p(0) + p(2)\,t - p(0)\,t^2 - p(2)\,t^3 = -6t + 6t^3 = -6\big(t - t^3\big).$$

     Thus $t - t^3$ is an eigenvector with eigenvalue $-6$.

Hence $\lambda = -6$, and its eigenspace is all multiples of $t - t^3$.

## Diagonal Matrix Representation

> **A matrix $A \in M_{n\times n}(\mathbb{C})$ (or $\mathbb{R}$) is diagonalizable if and only if there exists an invertible matrix $P$ and a diagonal matrix $D$ such that**
>
> $$A = P\,D\,P^{-1}.$$
>
> **Equivalently, for a linear operator $T$, if there exists a basis of eigenvectors for $V$, then the matrix representation of $T$ in that basis is a diagonal matrix.**

- **Key Idea**: Each eigenvalue $\lambda_i$ goes onto the diagonal of $D$, and its corresponding eigenvector becomes a column of $P$.

### Example: Find a Basis That Diagonalizes $A$

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(x) = Ax$, where

$$A = \begin{pmatrix} 0 & 1 \\ -3 & 4 \end{pmatrix}.$$

We want a basis $B$ of $\mathbb{R}^2$ such that $[T]_B$ is diagonal.

1. **Eigenvalues**: Solve $\det(A - \lambda I) = 0$.
2. **Eigenvectors**: Solve $(A - \lambda_i I)x = 0$ for each eigenvalue.
3. **Matrix $P$**: Formed by placing the independent eigenvectors as columns, in the same order used for the diagonal entries of $D$.
4. Then $P^{-1}AP$ is diagonal with eigenvalues on the diagonal.

## Summary and Key Points

- **Complex Eigenvalues** arise naturally if the characteristic polynomial has non-real roots.
- A **diagonalizable** matrix is one that can be written as $A = PDP^{-1}$ for some invertible $P$ and diagonal $D$.
- **Eigenspaces** are the sets of vectors scaled by each eigenvalue.
- In many real applications, we extend scalars to $\mathbb{C}$ to find *all* eigenvalues (Fundamental Theorem of Algebra).
- If you have fewer than $n$ linearly independent eigenvectors, the matrix is **not** diagonalizable. But over $\mathbb{C}$, one can still form its **Jordan Normal Form** if diagonalization fails.

> **In practice, to diagonalize $A$, you must find a complete set of linearly independent eigenvectors. If you cannot, the matrix is not diagonalizable. Over $\mathbb{C}$, you may still have complex eigenvalues and (if necessary) Jordan blocks.**

# 26. Preparation for Final Exam

## What This Note Is About

We have a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by the matrix

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix} \quad \text{so that} \quad T(\mathbf{x}) = A\mathbf{x}.$$

We work with respect to the basis

$$B = \{\, b_1, b_2 \,\} \quad \text{where} \quad b_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

**Objectives**:

1. **Show** that $b_1$ is an eigenvector of $A$.
2. **Prove** $A$ is **not** diagonalizable despite having an eigenvector $b_1$.
3. **Find** the matrix of $T$ in the basis $B$, denoted $[T]_B$.
4. **Include** a theorem on how to compute $[\, T(\mathbf{x}) \,]_B$ using $[T]_B$.

Below, we demonstrate **all solutions**, checking every condition for validity and correctness.

## Definition: When Is a Matrix Diagonalizable?

> **A square matrix** $A$ is **diagonalizable** if it has a basis consisting entirely of its eigenvectors. Equivalently, each eigenvalue's algebraic multiplicity must equal its geometric multiplicity.

## Step 1: Confirm $b_1$ is an Eigenvector

Let us first see why $b_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector of $A$.

1. **Eigenvalues**

   We calculate the characteristic polynomial of $A$:

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ -1 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) - (-1)\cdot 1 = (3 - 4\lambda + \lambda^2) + 1 = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2.$$

This shows the **only** eigenvalue is $\lambda = 2$, with **algebraic multiplicity** $2$.

2. **Eigenvectors**

   We solve $(A - 2I)\mathbf{x} = 0$. That is:

$$A - 2I = \begin{pmatrix} 1-2 & 1 \\ -1 & 3-2 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}.$$

   Row reduction gives one free variable:

$$\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \implies x_1 - x_2 = 0 \implies x_1 = x_2.$$

   Thus every eigenvector is of the form $t \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ for $t \neq 0$.

3. **Conclusion**

   Clearly, $b_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ lies in that eigenspace; hence it is an eigenvector for the eigenvalue $\lambda = 2$.

## Step 2: Show $A$ Is Not Diagonalizable

Since the characteristic polynomial is $(\lambda - 2)^2$, the algebraic multiplicity of $\lambda = 2$ is $2$. However, the eigenspace is spanned by $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ alone, giving a **one-dimensional** eigenspace.

**Remark:**

**Algebraic multiplicity = 2, but geometric multiplicity = 1.** Hence no second linearly independent eigenvector exists, and we cannot form a basis of $\mathbb{R}^2$ consisting **entirely** of eigenvectors. Therefore,

$$\text{matrix } A \text{ is NOT diagonalizable.}$$

## Step 3: Find $[T]_B$

> **Definition**
>
> If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is represented by $A$ in the standard basis, then its matrix in a different basis $B$ is defined by
>
> $$[T]_B = \big([T(b_1)]_B \quad [T(b_2)]_B\big),$$
>
> where $[v]_B$ denotes the coordinate vector of $v$ relative to basis $B$.

### (a) Compute $T(b_1)$ and its $B$-Coordinates

1. **Apply** $A$:

$$b_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad T(b_1) = A\, b_1 = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}.$$

2. **Express** $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ **as a combination of** $b_1$ **and** $b_2$:

$$\begin{pmatrix}2\\2\end{pmatrix} = \alpha\begin{pmatrix}1\\1\end{pmatrix} + \beta\begin{pmatrix}3\\4\end{pmatrix} = \begin{pmatrix}\alpha+3\beta\\\alpha+4\beta\end{pmatrix}.$$

Matching coordinates: $\alpha + 3\beta = 2$ and $\alpha + 4\beta = 2$. Subtract the first from the second:

$$(\alpha + 4\beta) - (\alpha + 3\beta) = 2 - 2 \implies \beta = 0.$$

Then $\alpha = 2$. So:

$$\begin{pmatrix}2\\2\end{pmatrix} = 2\,b_1 + 0\,b_2.$$

Hence,

$$\left[T(b_1)\right]_B = \begin{pmatrix}2\\0\end{pmatrix}.$$

### (b) Compute $T(b_2)$ and its $B$-Coordinates

1. **Apply $A$:**

$$b_2 = \begin{pmatrix}3\\4\end{pmatrix}, \quad T(b_2) = A\,b_2 = \begin{pmatrix}1 & 1\\-1 & 3\end{pmatrix}\begin{pmatrix}3\\4\end{pmatrix} = \begin{pmatrix}7\\9\end{pmatrix}.$$

2. **Express $\begin{pmatrix}7\\9\end{pmatrix}$ as a combination of $b_1$ and $b_2$:**

$$\begin{pmatrix}7\\9\end{pmatrix} = \alpha\begin{pmatrix}1\\1\end{pmatrix} + \beta\begin{pmatrix}3\\4\end{pmatrix} = \begin{pmatrix}\alpha+3\beta\\\alpha+4\beta\end{pmatrix}.$$

From $\alpha + 3\beta = 7$ and $\alpha + 4\beta = 9$, subtracting yields:

$$(\alpha + 4\beta) - (\alpha + 3\beta) = 9 - 7 \implies \beta = 2.$$

Then $\alpha + 6 = 7 \Rightarrow \alpha = 1$. Therefore:

$$\begin{pmatrix}7\\9\end{pmatrix} = 1\,b_1 + 2\,b_2.$$

Hence,

$$\left[T(b_2)\right]_B = \begin{pmatrix}1\\2\end{pmatrix}.$$

### (c) Form the Matrix $\left[T\right]_B$

We place $\left[T(b_1)\right]_B$ and $\left[T(b_2)\right]_B$ as columns:

$$[T]_B \begin{pmatrix}2 & 1\\0 & 2\end{pmatrix}$$

## Step 4: Theorem on B-Coordinates of $T(\mathbf{x})$

> **Theorem (Coordinate Transformation and Matrix Representation)**

> **Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map, and let $B$ be any basis of $\mathbb{R}^2$. If $\mathbf{x}$ is a vector in $\mathbb{R}^2$ with coordinates $[\mathbf{x}]_B$ relative to $B$, then**
>
> $$[T(\mathbf{x})]_B = [T]_B\,[\mathbf{x}]_B.$$
>
> **In other words, to find the coordinates of $T(\mathbf{x})$ in the basis $B$, multiply the matrix $[T]_B$ by the coordinate vector $[\mathbf{x}]_B$. This principle underlies how we "translate" a linear transformation between different bases.**

**Explanation**:

1. We start with an arbitrary vector $\mathbf{x}$ in $\mathbb{R}^2$.
2. We represent $\mathbf{x}$ in basis $B$ by the coordinate vector $[\mathbf{x}]_B$.
3. Applying $T$ to $\mathbf{x}$ in the standard basis is $T(\mathbf{x})$.
4. However, to express $T(\mathbf{x})$ again in $B$-coordinates, we use exactly $[T]_B \cdot [\mathbf{x}]_B$.

This gives a direct way to compute "what $T$ does" to any vector $\mathbf{x}$ in the language of the basis $B$.

## Summary of Results

1. **Eigenvector Check and Non-Diagonalizability**
   - The only eigenvalue of $A$ is $\lambda = 2$, and the eigenspace is spanned by $\begin{pmatrix}1\\1\end{pmatrix}$.
   - Since the algebraic multiplicity (2) exceeds the geometric multiplicity (1), $A$ **is not diagonalizable**.
2. **Matrix $[T]_B$**
   - We calculated
     $\big[T(b_1)\big]_B = \begin{pmatrix}2\\0\end{pmatrix}$,
     $\big[T(b_2)\big]_B = \begin{pmatrix}1\\2\end{pmatrix}$.
   - Hence,

$$[T]_B = \begin{pmatrix}2 & 1\\0 & 2\end{pmatrix}.$$

3. **B-Coordinate Transformation**
   - For **any $\mathbf{x}$**, the coordinates of $T(\mathbf{x})$ in basis $B$ follow from $[T(\mathbf{x})]_B = [T]_B\,[\mathbf{x}]_B$.

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