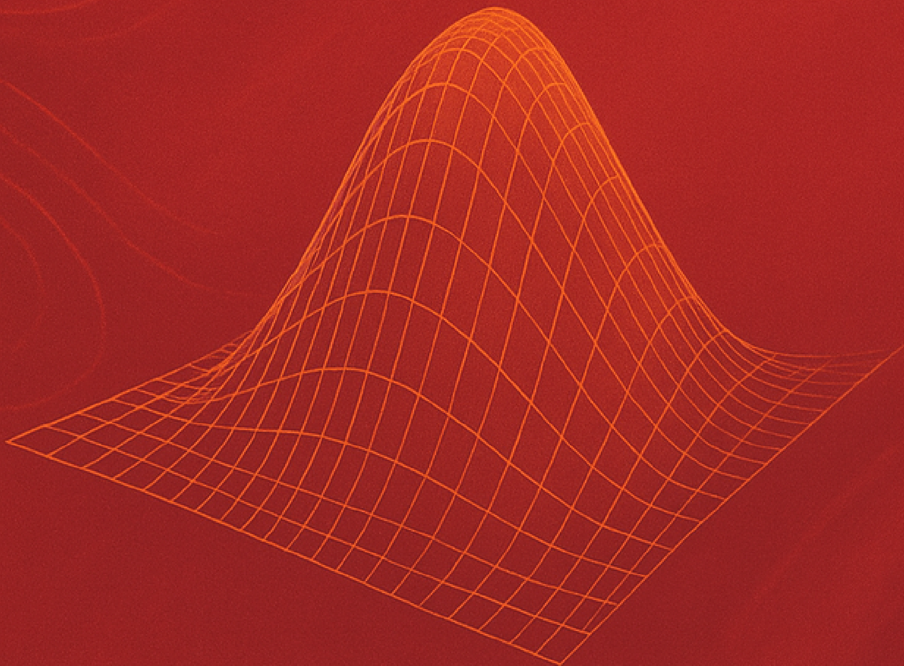

MATH203

COURSE NOTES

KOÇ UNIVERSITY

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This work is a personal academic compilation created for educational purposes as part of the **MATH203 (Multivariable Calculus)** course at **Koç University**.

Compiled in **Istanbul, Turkey**.

MATH203

1. Vectors and Coordinate Geometry in 3-Space

In our physical world, space is three-dimensional. This means that to locate any point in space, we require three numbers—one for each dimension. These concepts are fundamental in fields such as physics, engineering, and computer science. In this note, we will explore the basics of coordinate geometry in three dimensions, including how points are represented, how distances are calculated, and how these ideas extend to higher dimensions.

The 3-Dimensional World and Coordinate Systems

The physical world we experience is 3-dimensional. At any given point, we can define three mutually perpendicular directions. In linear algebra and geometry, these directions are typically represented by the three coordinate axes: x, y, and z.

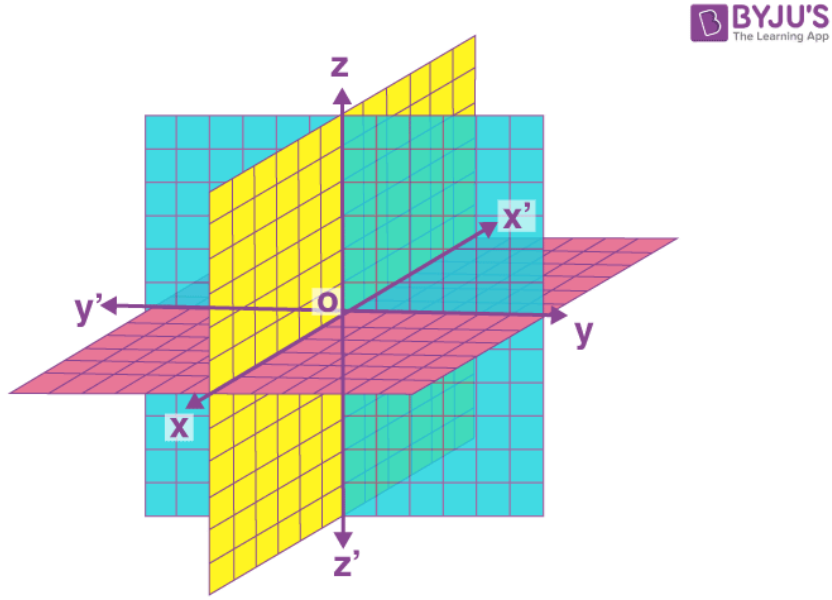
> **QUANTITIES:** A vector is a quantity with both magnitude and direction, while a scalar is a quantity with only magnitude.

To uniquely identify any point in 3-dimensional space, we use an ordered triple of real numbers. For example, a point P is written as:

$$P = (x, y, z)$$

This space is denoted by $\mathbb{R}^3$, where:

$$\mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$$



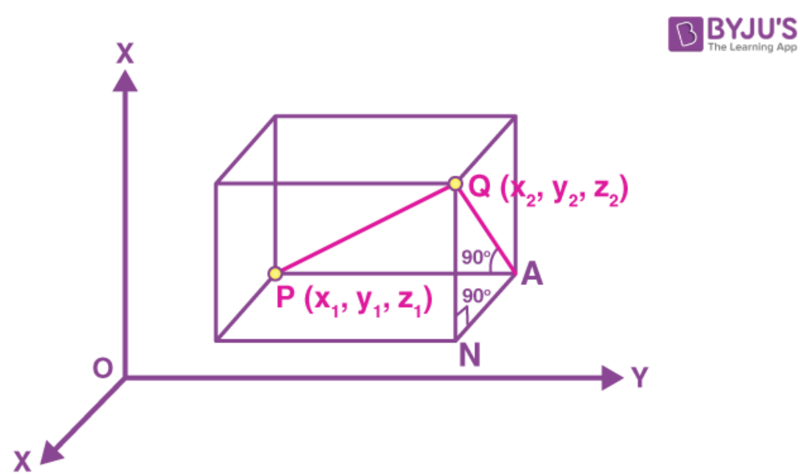
Distance in $\mathbb{R}^3$

The distance between two points in 3-dimensional space is a direct extension of the Pythagorean theorem. For a point $P = (x, y, z)$ and the origin $O = (0, 0, 0)$, the distance r from the origin to P is given by:

$$r = \sqrt{x^2 + y^2 + z^2}$$

Similarly, the distance between two arbitrary points $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ is:

$$d(P_1, P_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$



Example: Right Triangle in 3-Space

Consider a triangle with vertices:

$$A = (1, -1, 2), \quad B = (3, 3, 8), \quad C = (2, 0, 1)$$

To determine if this triangle has a right angle, we calculate the lengths of its sides.

Step 1: Compute Side Lengths

- **Side AB:**

$$|AB| = \sqrt{(3-1)^2 + (3-(-1))^2 + (8-2)^2} = \sqrt{2^2 + 4^2 + 6^2} = \sqrt{4+16+36} = \sqrt{56}$$

- **Side AC:**

$$|AC| = \sqrt{(2-1)^2 + (0-(-1))^2 + (1-2)^2} = \sqrt{1^2 + 1^2 + (-1)^2} = \sqrt{1+1+1} = \sqrt{3}$$

- **Side BC:**

$$|BC| = \sqrt{(3-2)^2 + (3-0)^2 + (8-1)^2} = \sqrt{1^2 + 3^2 + 7^2} = \sqrt{1+9+49} = \sqrt{59}$$

Step 2: Verify the Pythagorean Theorem

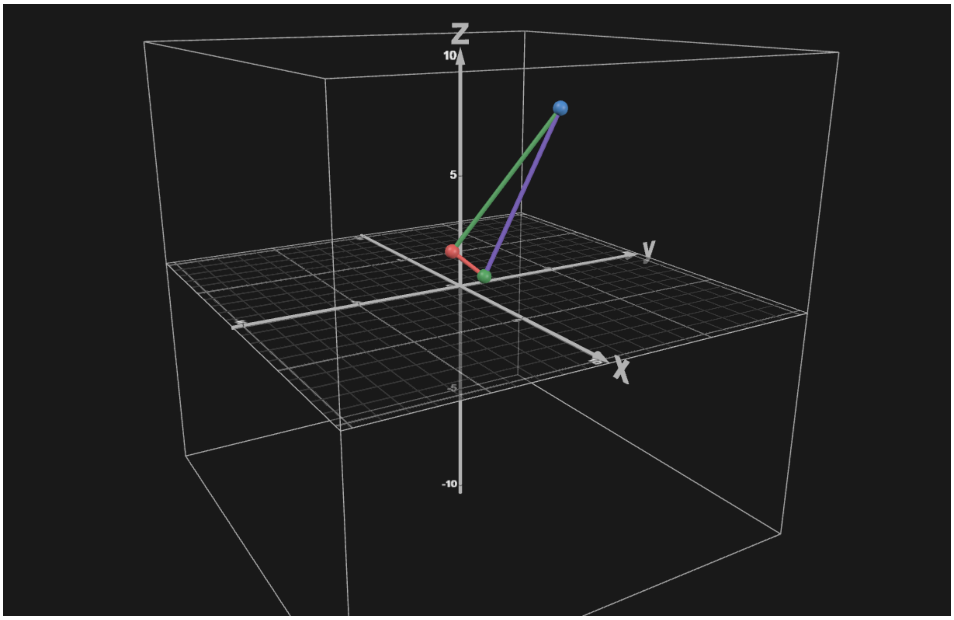
For a triangle with a right angle at A, the lengths must satisfy:

$$|AB|^2 + |AC|^2 = |BC|^2$$

Plug in the values:

$$56 + 3 = 59$$

Since the equality holds, the triangle is right-angled at A.



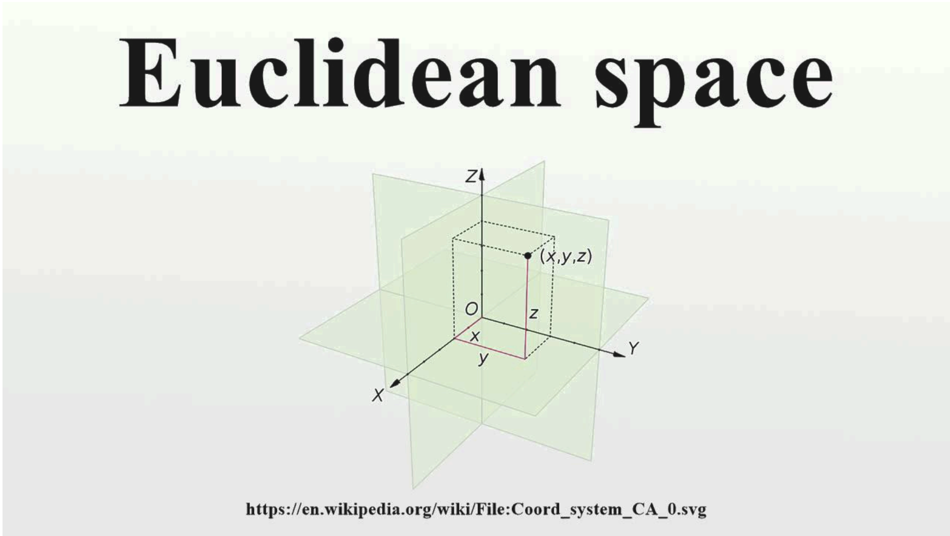
Euclidean n-Space

The ideas from 3-dimensional space extend naturally to higher dimensions. For any positive integer n, the Euclidean space $\mathbb{R}^n$ is defined as:

$$\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{R},\ i = 1, 2, \ldots, n\}$$

The distance between two points $P_1 = (x_1, x_2, \ldots, x_n)$ and $P_2 = (y_1, y_2, \ldots, y_n)$ in $\mathbb{R}^n$ is given by:

$$d(P_1, P_2) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}$$



Additional Information and Examples

Importance of Coordinate Geometry

Coordinate geometry provides the framework for analyzing spatial relationships using algebra. This approach is fundamental in many fields:

- In physics, to describe the motion of objects.
- In engineering, for designing structures and systems.
- In computer graphics, for rendering scenes in 3D.

Applications of Euclidean Distance

Understanding the distance formula in $\mathbb{R}^3$ and $\mathbb{R}^n$ is crucial for tasks such as:

- **Navigation:** Calculating the straight-line distance between two locations.
- **Data Analysis:** Computing distances in high-dimensional spaces for clustering algorithms.
- **Robotics:** Determining how far a robot must move to reach a target point.

Example: Distance in $\mathbb{R}^4$

Consider points in four-dimensional space:

$$P_1 = (1, 2, 3, 4) \quad \text{and} \quad P_2 = (4, 3, 2, 1)$$

Their distance is:

$$d(P_1, P_2) = \sqrt{(1-4)^2 + (2-3)^2 + (3-2)^2 + (4-1)^2} = \sqrt{(-3)^2 + (-1)^2 + (1)^2 + (3)^2} = \sqrt{9+1+1+9} = \sqrt{20}$$

Summary

In this lecture note, we have covered:

- **Vectors and Coordinate Geometry in 3-Space:** The physical world is three-dimensional, and any point in space is represented by an ordered triple of real numbers.
- **Coordinates in $\mathbb{R}^3$:** Points are denoted as $P = (x, y, z)$ in $\mathbb{R}^3$, where $\mathbb{R}^3$ is defined as $\{(x, y, z) \mid x, y, z \in \mathbb{R}\}$.

- **Distance in $\mathbb{R}^3$:** The distance from the origin to a point $P = (x, y, z)$ is computed using $r = \sqrt{x^2 + y^2 + z^2}$, and the distance between two points follows the generalized Pythagorean theorem.
- **Euclidean n-Space:** Extended the concept of coordinate geometry to n dimensions with the space $\mathbb{R}^n$ and the corresponding distance formula.

Understanding these concepts lays the foundation for more advanced studies in calculus, physics, and engineering, where precise spatial measurements and relationships are crucial.

Self Test

Self-Test: Lecture 1

Raw Notes

Raw Notes

2. Vectors and Coordinate Geometry in 3-Space - Extended

Analytic Geometry in Three Dimensions

Our physical space is three-dimensional. To uniquely locate any point in this space, we require three coordinates, one for each mutually perpendicular axis. In linear algebra and geometry, these axes are typically denoted as x, y, and z.

> **COORDINATE SYSTEM:** A framework that uses an ordered triple (x, y, z) to specify the position of a point in $\mathbb{R}^3$.

The three-dimensional space is defined as:

$$\mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$$

A point P in this space is represented by:

$$P = (x, y, z)$$

Distance in $\mathbb{R}^3$

The distance from the origin $O = (0, 0, 0)$ to a point $P = (x, y, z)$ is calculated by:

$$r = \sqrt{x^2 + y^2 + z^2}$$

Similarly, the distance between any two points $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ is:

$$d(P_1, P_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

Euclidean n-Space

The concept of coordinates extends beyond three dimensions. For any positive integer n, the Euclidean space $\mathbb{R}^n$ is defined as:

$$\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{R},\ i = 1, 2, \ldots, n\}$$

The distance between two points $P_1 = (x_1, x_2, \ldots, x_n)$ and $P_2 = (y_1, y_2, \ldots, y_n)$ in $\mathbb{R}^n$ is:

$$d(P_1, P_2) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}$$

Introduction to Vectors

Vectors are quantities that have both magnitude and direction, distinguishing them from scalars (which have only magnitude). They are essential in describing many physical phenomena such as displacement, velocity, and force.

> **VECTOR:** A quantity with both magnitude and direction, represented as:
>
> $$\mathbf{A} = |\mathbf{A}|\,\hat{A}$$
>
> where $|\mathbf{A}|$ is the magnitude and $\hat{A}$ is the unit vector indicating direction.

Unlike fixed points, vectors can be “moved” (translated) without changing their intrinsic properties.

Unit Vectors and Standard Basis Vectors

Unit Vectors

A unit vector has a magnitude of 1 and indicates direction. It is used to express the direction component of any vector without affecting its magnitude.

Standard Basis Vectors

In three-dimensional space, the standard basis vectors are:

$$\mathbf{i} = (1, 0, 0), \quad \mathbf{j} = (0, 1, 0), \quad \mathbf{k} = (0, 0, 1)$$

Any vector $\mathbf{A}$ in $\mathbb{R}^3$ can be expressed as:

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$$

These standard basis vectors provide a foundation for representing any vector in component form.

Vector Operations: Addition, Subtraction, and Scalar Multiplication

Vector Addition and Subtraction

Vectors are added by combining their corresponding components:

$$\mathbf{A} + \mathbf{B} = (A_x + B_x)\mathbf{i} + (A_y + B_y)\mathbf{j} + (A_z + B_z)\mathbf{k}$$

Properties:

- **Commutative:** $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$
- **Associative:** $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$

Subtraction is defined as the addition of the negative:

$$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B})$$

where $-\mathbf{B}$ reverses the direction of $\mathbf{B}$.

Scalar Multiplication

When a vector is multiplied by a scalar c, its magnitude is scaled by $|c|$ while its direction remains unchanged (if $c > 0$) or reverses (if $c < 0$):

$$c\mathbf{A} = (cA_x)\mathbf{i} + (cA_y)\mathbf{j} + (cA_z)\mathbf{k}$$

If $c = 0$, the result is the zero vector, $\mathbf{0}$.

Components, Magnitude, and Direction of a Vector

Components and Vector Notation

Any vector in $\mathbb{R}^2$ can be written as:

$$\mathbf{R} = R_x\mathbf{i} + R_y\mathbf{j}$$

And in $\mathbb{R}^3$:

$$\mathbf{R} = R_x\mathbf{i} + R_y\mathbf{j} + R_z\mathbf{k}$$

This representation is essential for performing arithmetic operations on vectors.

Calculating Magnitude

The magnitude of a vector $\mathbf{R}$ in $\mathbb{R}^3$ is:

$$|\mathbf{R}| = \sqrt{R_x^2 + R_y^2 + R_z^2}$$

In $\mathbb{R}^2$, the formula simplifies to:

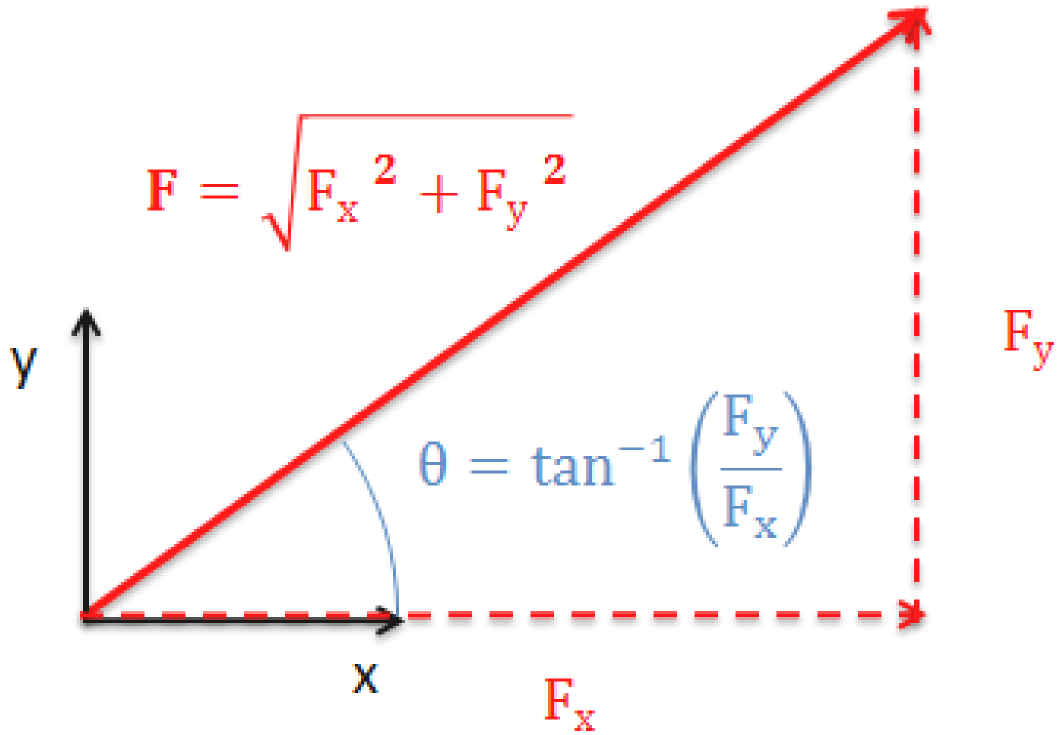
$$|\mathbf{R}| = \sqrt{R_x^2 + R_y^2}$$

Determining Direction

For a vector in two dimensions, the angle θ (with respect to the x-axis) is given by:

$$\theta = \arctan\left(\frac{R_y}{R_x}\right)$$

In three dimensions, direction is typically represented by the unit vector, though spherical coordinates may also be used.



Dot Product (Scalar Product)

The dot product of two vectors $\mathbf{A}$ and $\mathbf{B}$ is defined as:

$$\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z = |\mathbf{A}|\,|\mathbf{B}| \cos\theta$$

where θ is the angle between the vectors.

Key Points:

- It results in a scalar.
- **Commutative:** $\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$
- It measures the extent to which one vector projects onto another.

Example:

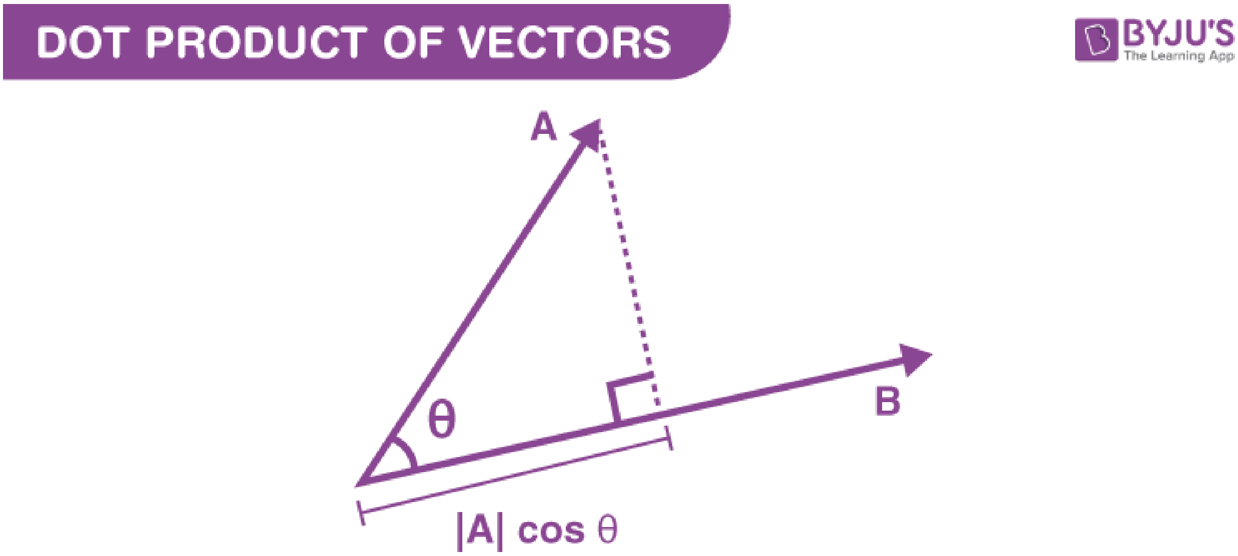
Let $\mathbf{A} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ and $\mathbf{B} = -4\mathbf{i} + 2\mathbf{j} - \mathbf{k}$. Then:

$$\mathbf{A} \cdot \mathbf{B} = (2 \times -4) + (3 \times 2) + (1 \times -1) = -8 + 6 - 1 = -3$$

The angle θ can be found by:

$$\theta = \arccos\left(\frac{-3}{|\mathbf{A}|\,|\mathbf{B}|}\right)$$

Dot Product	Result
$\mathbf{i} \cdot \mathbf{i}$	1
$\mathbf{j} \cdot \mathbf{j}$	1
$\mathbf{k} \cdot \mathbf{k}$	1
$\mathbf{i} \cdot \mathbf{j}$	0
$\mathbf{j} \cdot \mathbf{k}$	0
$\mathbf{k} \cdot \mathbf{i}$	0



Cross Product (Vector Product)

The cross product of two vectors in $\mathbb{R}^3$ results in a vector that is perpendicular to both:

$$\mathbf{A} \times \mathbf{B} = (A_yB_z - A_zB_y)\mathbf{i} + (A_zB_x - A_xB_z)\mathbf{j} + (A_xB_y - A_yB_x)\mathbf{k}$$

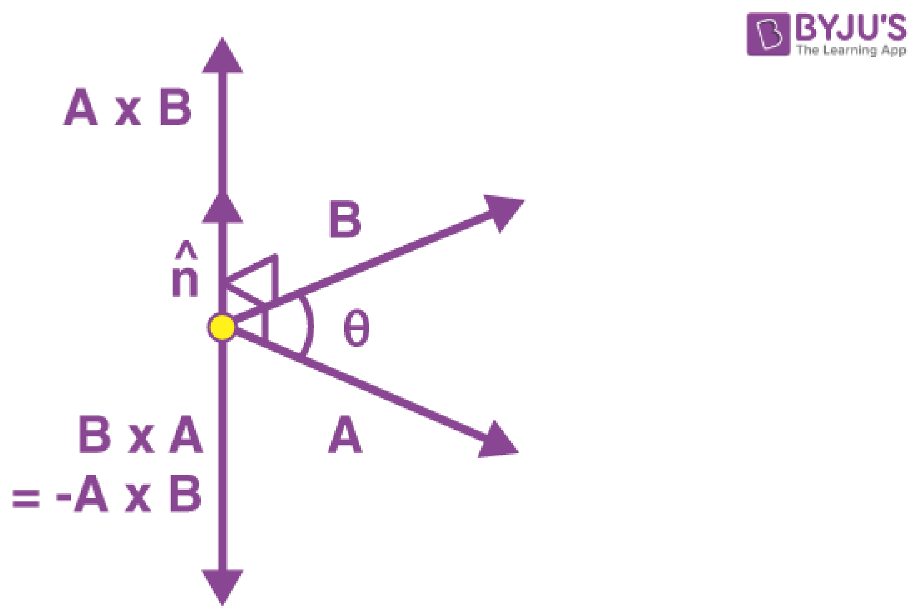
The magnitude of the cross product is:

$$|\mathbf{A} \times \mathbf{B}| = |\mathbf{A}|\,|\mathbf{B}|\sin\theta$$

where θ is the angle between $\mathbf{A}$ and $\mathbf{B}$.

Properties:

- **Anticommutative:** $\mathbf{A} \times \mathbf{B} = -(\mathbf{B} \times \mathbf{A})$
- The resulting vector is orthogonal to both $\mathbf{A}$ and $\mathbf{B}$.
- If $\mathbf{A}$ and $\mathbf{B}$ are parallel, then $\mathbf{A} \times \mathbf{B} = \mathbf{0}$.



Cross Product of Unit Vectors

The cross product is an operation on two vectors in three-dimensional space that results in a vector perpendicular to both. For the standard basis vectors in $\mathbb{R}^3$, the following relationships hold:

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}$$

$$\mathbf{j} \times \mathbf{k} = \mathbf{i}$$

$$\mathbf{k} \times \mathbf{i} = \mathbf{j}$$

These rules demonstrate the **anticommutative** property of the cross product:

$$\mathbf{j} \times \mathbf{i} = -\mathbf{k}, \quad \mathbf{k} \times \mathbf{j} = -\mathbf{i}, \quad \mathbf{i} \times \mathbf{k} = -\mathbf{j}$$

This operation is fundamental in determining directions perpendicular to a given plane, with applications in physics, computer graphics, and engineering.

Cross Product	Result
$\mathbf{i} \times \mathbf{i}$	$\mathbf{0}$
$\mathbf{j} \times \mathbf{j}$	$\mathbf{0}$
$\mathbf{k} \times \mathbf{k}$	$\mathbf{0}$
$\mathbf{i} \times \mathbf{j}$	$\mathbf{k}$
$\mathbf{j} \times \mathbf{k}$	$\mathbf{i}$
$\mathbf{k} \times \mathbf{i}$	$\mathbf{j}$
$\mathbf{j} \times \mathbf{i}$	$-\mathbf{k}$
$\mathbf{k} \times \mathbf{j}$	$-\mathbf{i}$
$\mathbf{i} \times \mathbf{k}$	$-\mathbf{j}$

Cross Product as a Determinant

The cross product of two vectors in $\mathbb{R}^3$ can be computed using the determinant of a 3×3 matrix. Given two vectors:

$$\mathbf{A} = (A_x, A_y, A_z) \quad \text{and} \quad \mathbf{B} = (B_x, B_y, B_z),$$

their cross product $\mathbf{A} \times \mathbf{B}$ is given by the determinant:

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \mathbf{i}(A_y B_z - A_z B_y) - \mathbf{j}(A_x B_z - A_z B_x) + \mathbf{k}(A_x B_y - A_y B_x).$$

This determinant method neatly encapsulates the computation and clearly shows that the resulting vector is perpendicular to both $\mathbf{A}$ and $\mathbf{B}$.

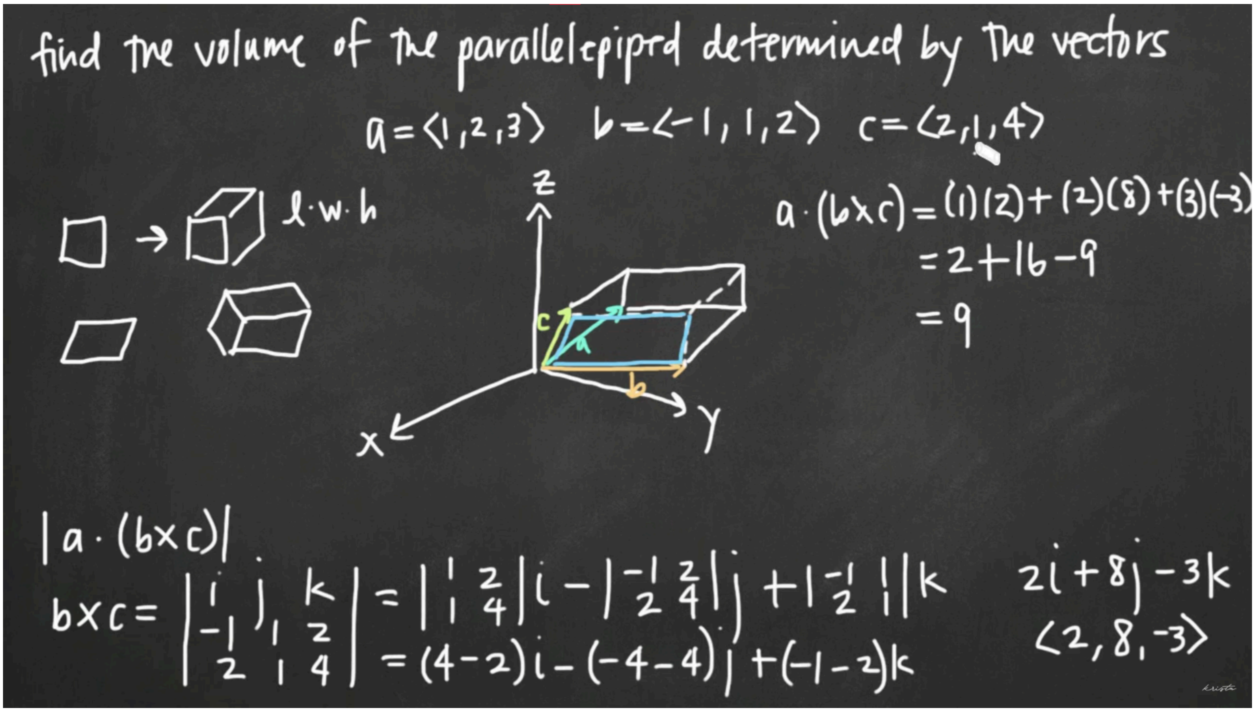
Scalar Triple Product (Advanced Topic)

The scalar triple product of three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ is defined as:

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$$

This product gives the volume of the parallelepiped spanned by the three vectors. If the result is zero, the vectors are coplanar.

> **SCALAR TRIPLE PRODUCT:** A measure of the volume of a parallelepiped. A zero result indicates the vectors lie in the same plane.



Summary of Vector Concepts

- **Definition and Representation:** Vectors have both magnitude and direction and are represented as $\mathbf{A} = |\mathbf{A}|\,\hat{A}$ or in component form.
- **Unit and Standard Basis Vectors:** Unit vectors (with magnitude 1) define direction. In $\mathbb{R}^3$, standard basis vectors are $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.
- **Vector Operations:**
  - **Addition/Subtraction:** Performed component-wise; subtraction is adding the negative.
  - **Scalar Multiplication:** Scales the vector's magnitude.
- **Magnitude and Direction:** The magnitude is found using the Euclidean norm, and the direction can be determined using trigonometric functions (e.g., $\arctan$ for 2D).
- **Dot Product:** Provides a scalar indicating how much one vector extends in the direction of another.
- **Cross Product:** Yields a vector perpendicular to two given vectors; its magnitude is related to the sine of the angle between them.
- **Scalar Triple Product:** Gives the volume of a parallelepiped formed by three vectors and indicates coplanarity when zero.

Understanding these concepts is essential for applications in physics, computer graphics, engineering, and advanced mathematics.

Self Test

Self-Test: Lecture 2

Raw Notes

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3. Planes and Lines in 3-Space

Planes in 3-Space

> **PLANE (Point-Normal Form):** A plane in three-dimensional space can be uniquely defined by a point $P_0 = (x_0, y_0, z_0)$ that lies on the plane and a nonzero normal vector $\mathbf{n} = (a, b, c)$. The plane consists of all points $P = (x, y, z)$ whose position vectors $\mathbf{r}$ satisfy
>
> $$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$$

In scalar form, this equation becomes:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

Examples:

1. **Example (i):**

   The equation

$$2x - by - hz = 0$$

represents a plane passing through the origin $(0, 0, 0)$ with normal vector $(2, -b, -h)$. (Here, specific values for b and h are needed for a concrete example; assume $b = 3$ and $h = 4$ for instance, then the normal vector would be $(2, -3, -4)$.)

2. **Example (ii):**

   The plane given by

   $$2x + y + 3z = 6$$

   passes through a point such as $(3, 0, 0)$ and has a normal vector $(2, 1, 3)$.

3. **Example (iii):**

   The equation

   $$2x - y = 0$$

   represents a plane that is vertical (parallel to the z-axis) and has a normal vector $(2, -1, 0)$.

4. **Example (iv):**

   To find the plane passing through the point $(2, 0, 1)$ and perpendicular to the line passing through $(1, 1, 0)$ and $(4, -1, -2)$, first compute the direction vector of the line:

   $$\mathbf{v} = (4 - 1,\ -1 - 1,\ -2 - 0) = (3, -2, -2).$$

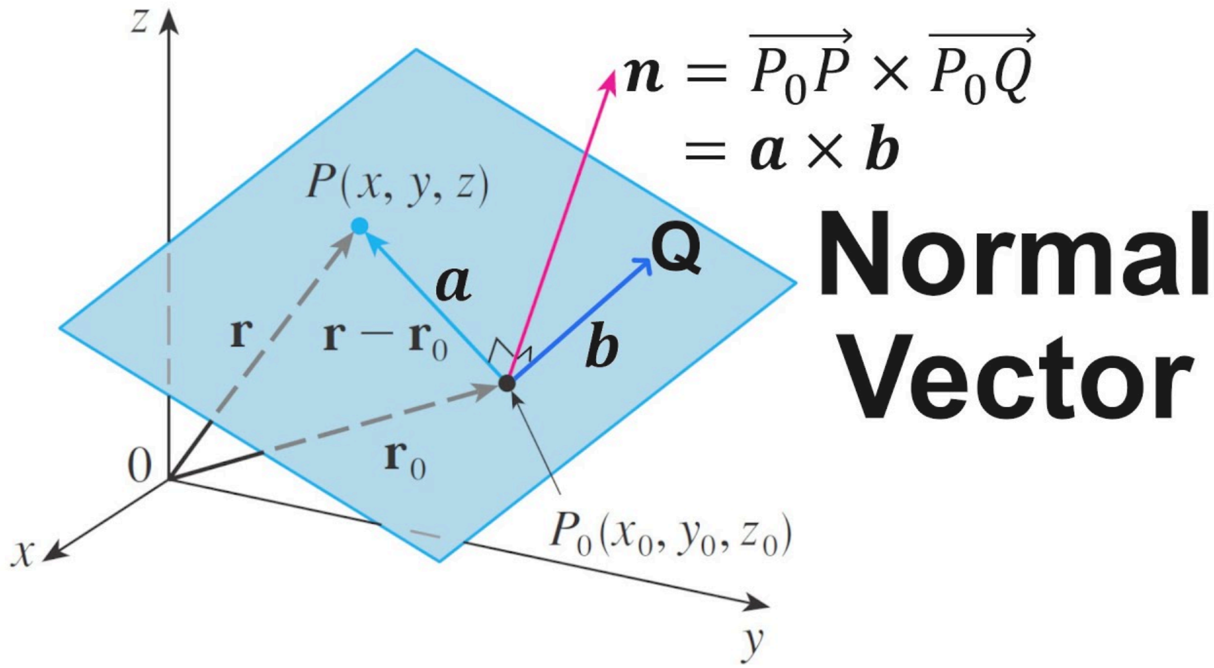
   The plane's normal vector is parallel to $\mathbf{v}$. Thus, using the point-normal form:

   $$3(x - 2) - 2(y - 0) - 2(z - 1) = 0 \quad \Longrightarrow \quad 3x - 2y - 2z = 4.$$

5. **Example (v):**

   A plane with intercepts a, b, and c on the x, y, and z axes respectively can be expressed as:

   $$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$



Lines in 3-Space

> **LINE (Parametric Form):** A line in three-dimensional space is determined by a point $P_0 = (x_0, y_0, z_0)$ through which the line passes and a nonzero direction vector $\mathbf{v} = (a, b, c)$. The line is given by the parametric equation:
>
> $$\mathbf{r}(t) = \mathbf{r}_0 + t\,\mathbf{v}$$
>
> where t is a real parameter.

This equation indicates that every point on the line can be reached by starting at P_0 and moving some scalar multiple t of the direction vector $\mathbf{v}$.

Examples:

1. **Example:**

   Consider the line defined by:

   $$x = 1 + t, \quad y = -3, \quad z = 4t.$$

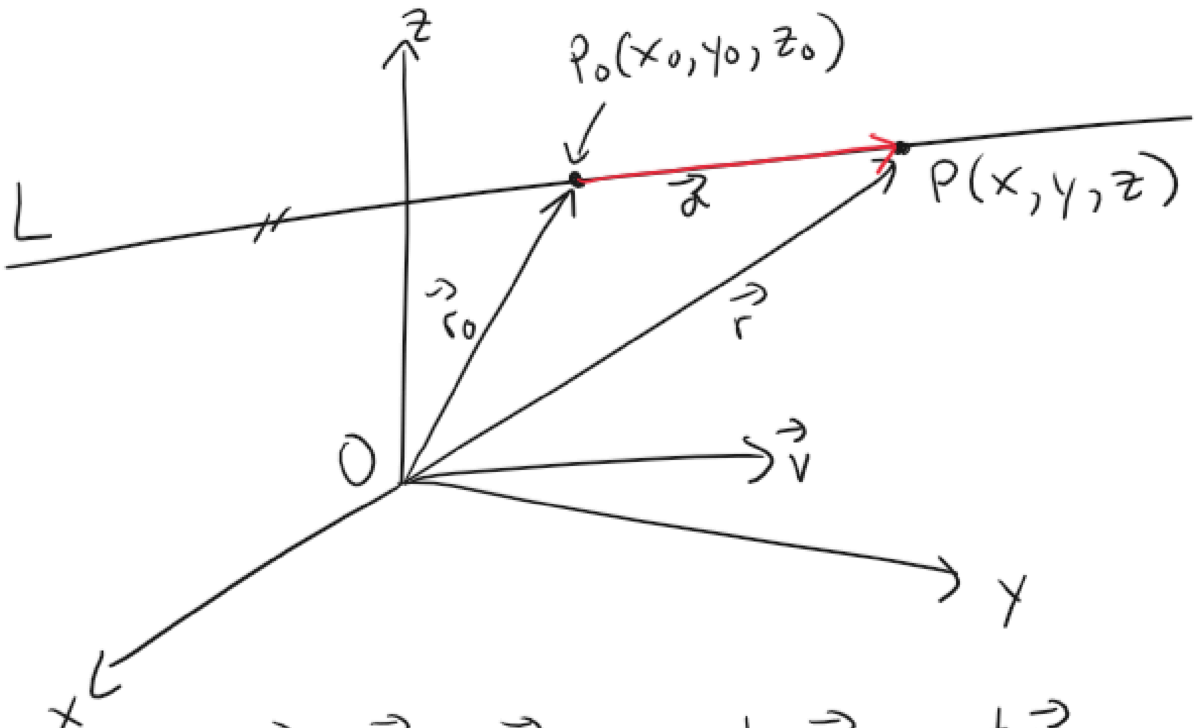
   Here, the line passes through the point $(1, -3, 0)$ (when $t = 0$) and has the direction vector $(1, 0, 4)$.

2. **Additional Example:**

   Another line can be represented as:

   $$\mathbf{r}(t) = (2, 1, 5) + t\,(3, -2, 1),$$

   which indicates that the line passes through $(2, 1, 5)$ and is parallel to the vector $(3, -2, 1)$.



$$\vec{r} = \vec{r}_0 + \vec{a} \qquad \text{and} \qquad \vec{a} = t\vec{v}$$

$$\rightarrow \boxed{\vec{r} = \vec{r}_0 + t\vec{v}}$$

Summary

In this study material, we have covered:

- **Planes in 3-Space:**
  - A plane is defined by a point and a normal vector.

  - Its equation can be written in vector form $\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$ or in scalar form $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$.
- **Lines in 3-Space:**
  - A line is represented by a point and a direction vector.
  - The parametric form $\mathbf{r}(t) = \mathbf{r}_0 + t\,\mathbf{v}$ describes all points on the line.

This material lays a foundation for further studies in multivariable calculus and analytic geometry, as well as applications in fields such as computer graphics, engineering, and physics.

Raw Notes

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4. Analytic Geometry in Three Dimensions

Representing Points in 3-Space

> **POSITION VECTOR:** A vector that specifies the location of a point in three-dimensional space.
>
> A point P is given by the ordered triple
>
> $$P = (x, y, z)$$
>
> and the set of all points in 3-space is defined as
>
> $$\mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}.$$

In linear algebra, any basis for $\mathbb{R}^3$ consists of three linearly independent vectors, which reflects the requirement of three numbers to locate a point.

Distance from a Point to the Origin

The distance r from a point $P = (x, y, z)$ to the origin $O = (0, 0, 0)$ is calculated using the 3-dimensional Pythagorean theorem:

$$r = \sqrt{x^2 + y^2 + z^2}.$$

This distance is equivalent to the magnitude of the position vector $\mathbf{r}$.

Distance Between Two Points

For any two points $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ in $\mathbb{R}^3$, the distance between them is given by:

$$d(P_1, P_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$$

This formula generalizes the Pythagorean theorem to three dimensions.

Example: Right Triangle in 3-Space

Consider a triangle with vertices:

- $A = (1, -1, 2)$
- $B = (3, 3, 8)$
- $C = (2, 0, 1)$

Step 1: Compute Side Lengths

- **Side AB:**

$$|AB| = \sqrt{(3-1)^2 + (3-(-1))^2 + (8-2)^2} = \sqrt{2^2 + 4^2 + 6^2} = \sqrt{4 + 16 + 36} = \sqrt{56}.$$

- **Side AC:**

$$|AC| = \sqrt{(2-1)^2 + (0-(-1))^2 + (1-2)^2} = \sqrt{1^2 + 1^2 + (-1)^2} = \sqrt{1+1+1} = \sqrt{3}.$$

- **Side BC:**

$$|BC| = \sqrt{(3-2)^2 + (3-0)^2 + (8-1)^2} = \sqrt{1^2 + 3^2 + 7^2} = \sqrt{1 + 9 + 49} = \sqrt{59}.$$

Step 2: Verify the Right Angle

If the triangle is right-angled at vertex A, then:

$$|BC|^2 = |AB|^2 + |AC|^2.$$

Substituting the values:

$$59 = 56 + 3 = 59.$$

Since the equality holds, the triangle has a right angle at A.

Equations of Surfaces in 3D

Digital geometry uses equations to describe various surfaces in space. Common examples include:

- **Plane:**

A plane passing through a point $P_0 = (x_0, y_0, z_0)$ with a nonzero normal vector $\mathbf{n} = (a, b, c)$ is defined by:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

Example: The equation $z = 0$ represents the horizontal (xy) plane.

- **Vertical Plane:**

  An equation like $x = y$ describes a vertical plane that contains the line $x = y$ in the xy-plane.

- **Cylinder:**

  A vertical circular cylinder with radius r is given by:

  $$x^2 + y^2 = r^2.$$

- **Sphere:**

  A sphere with center (h, k, l) and radius r is represented as:

  $$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2.$$

  Example: $x^2 + y^2 + z^2 = 36$ represents a sphere of radius 6 centered at the origin.

Euclidean n-Space

> **EUCLIDEAN n-SPACE:** For any positive integer n, the n-dimensional Euclidean space is defined as
>
> $$\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) \mid x_1, x_2, \ldots, x_n \in \mathbb{R}\}.$$
>
> The distance between two points $P_1 = (x_1, \ldots, x_n)$ and $P_2 = (y_1, \ldots, y_n)$ is given by
>
> $$d(P_1, P_2) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}.$$

This generalization allows us to extend the concepts of geometry in 3-space to higher dimensions.

Summary

In this section, we have covered:

- **Representation of Points:** How points in 3D are represented as ordered triples in $\mathbb{R}^3$.
- **Distance Calculations:** How to compute the distance from a point to the origin and between two points using the 3D extension of the Pythagorean theorem.
- **Equations of Surfaces:** Common forms for planes, vertical planes, cylinders, and spheres.
- **Euclidean n-Space:** The extension of these geometric concepts to n dimensions.

This material lays the groundwork for further studies in multivariable calculus and analytic geometry.

Raw Notes

Raw Notes

5. Curves and Parametrization

Curves and Parametrizations

Definition: Curve in $\mathbb{R}^3$

> **CURVE:** A curve in $\mathbb{R}^3$ can be regarded as the image of a vector-valued function
>
> $$\mathbf{r}(t) \;=\; x(t)\,\mathbf{i} \;+\; y(t)\,\mathbf{j} \;+\; z(t)\,\mathbf{k}, \quad a \leq t \leq b$$
>
> The function $\mathbf{r}(t)$ is often called a **parametrization** of the curve.

- Intuitively, as t varies from a to b, the point $\big(x(t), y(t), z(t)\big)$ traces out the curve in space.
- A curve can also exist in $\mathbb{R}^2$ when $z(t)$ is absent or fixed.

Examples of Parametrizations

Example 1: Line of Intersection of Two Planes

Suppose we have the planes $y = 2x - 4$ and $z = 3x + 1$. We want to parametrize the line of intersection from the point $(2, 0, 7)$ to $(3, 2, 10)$.

1. From $y = 2x - 4$, we set $y = t$ as our parameter. Then x can be solved as

$$t = 2x - 4 \quad\Longrightarrow\quad x = \frac{t+4}{2}$$

2. From $z = 3x + 1$, substitute $x = \frac{t+4}{2}$ to get

$$z = 3\left(\frac{t+4}{2}\right) + 1 = \frac{3t+12}{2} + 1 = \frac{3t+14}{2}$$

3. Hence the parametric equations become

$$x(t) = \frac{t+4}{2}, \quad y(t) = t, \quad z(t) = \frac{3t+14}{2}$$

4. Putting them together in vector form:

$$\mathbf{r}(t) = \left\langle \frac{t+4}{2},\ t,\ \frac{3t+14}{2} \right\rangle$$

5. We determine the parameter range $0 \le t \le 2$ to move from $(2, 0, 7) \to (3, 2, 10)$.

Example 2: Intersection of a Plane and a Paraboloid

- The plane is $x + y = 1$.
- The paraboloid is $z = x^2 + y^2$.
- One can parametrize this intersection by choosing:
    - $t = x$, so $y = 1 - t$, then $z = t^2 + (1-t)^2$.
    - Alternatively, $t = y$ or $t = z$. Each choice leads to a different but valid parametrization.
- In general, there can be multiple **equivalent** parametrizations for the same curve.

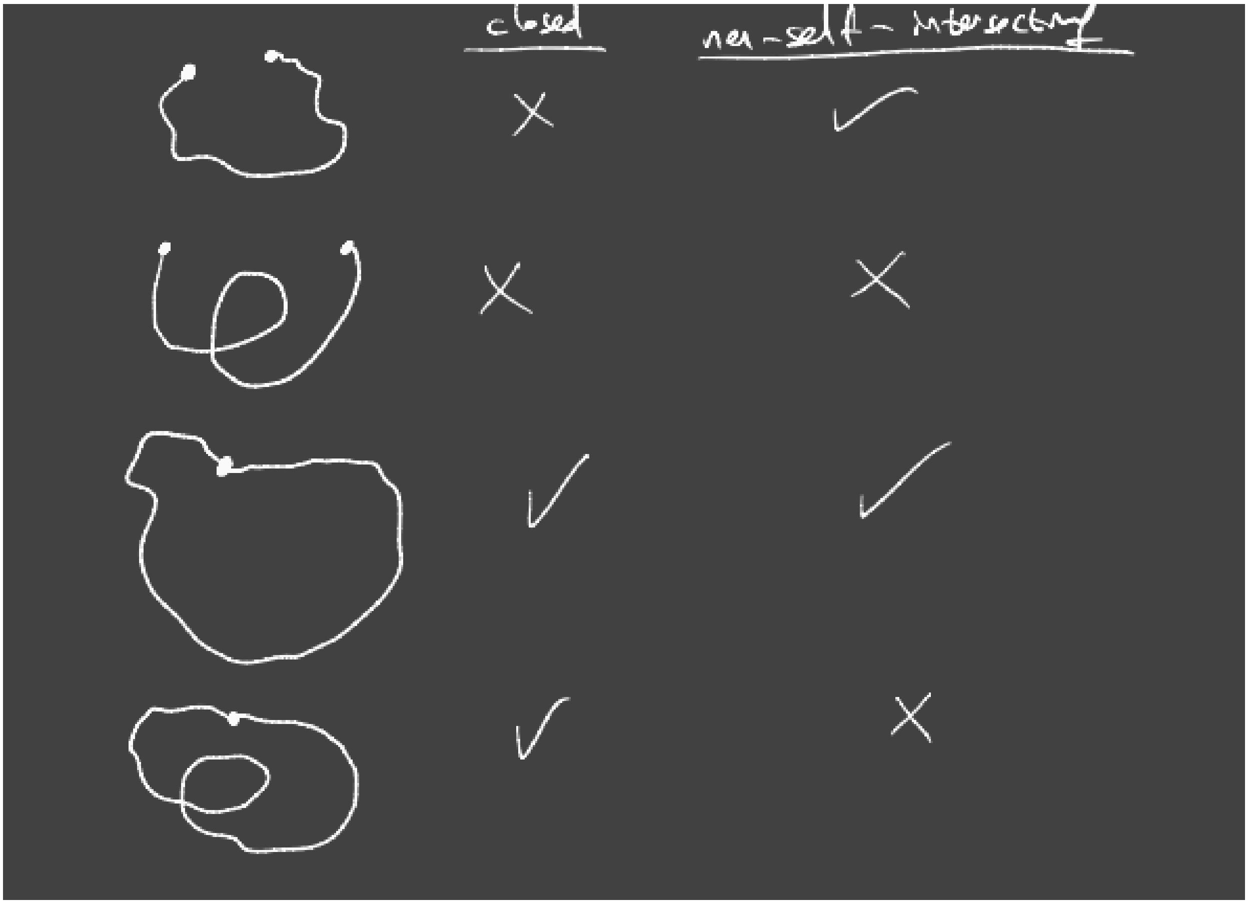
Example 3: 2D Parametrizations

- $\mathbf{r}(t) = \langle \sin t,\ \cos t \rangle,\ -\pi/2 \le t \le \pi/2.$
- $\mathbf{r}(t) = \langle t - 1,\ \sqrt{2t - t^2} \rangle,\ 0 \le t \le 2.$
- $\mathbf{r}(t) = \langle t\sqrt{2 - t^2},\ 1 - t^2 \rangle,\ -1 \le t \le 1.$

In several such examples, $|\mathbf{r}(t)| = 1$ or the resulting curve is a **semicircle** in the xy-plane from $(-1, 0)$ to $(1, 0)$.

Closed and Not-Self-Intersecting Curves

> **CLOSED CURVE:** A curve $\mathbf{r}(t)$, $a \le t \le b$, is closed if $\mathbf{r}(a) = \mathbf{r}(b)$.
>
> **NOT-SELF-INTERSECTING:** A curve is **not-self-intersecting** if its parametrization is **one-to-one** (injective) on $[a, b]$, except possibly for the endpoints if it is closed.



Smoothness

> **SMOOTH CURVE (of order 1)**: A curve $\mathbf{r}(t)$ is said to be smooth of order 1 if it has a continuous first derivative $\mathbf{r}'(t)$ on $[a, b]$. If it has derivatives of all orders, we call it a (fully) **smooth** curve.

- "Smoothness" of a curve measures the **continuity** of its **derivatives**.
- In most calculus or geometry contexts, "smooth" often means at least C^1: continuous and once differentiable with no corners.

Arc Length

Definition: Arc Length of a Smooth Curve

Let $\mathbf{r}(t)$ be a **bounded, continuous**, and **smooth** curve for $a \leq t \leq b$. We define the **arc length** S by the limit of polygonal approximations:

1. Subdivide $[a, b]$ into n subintervals:

$$a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$$

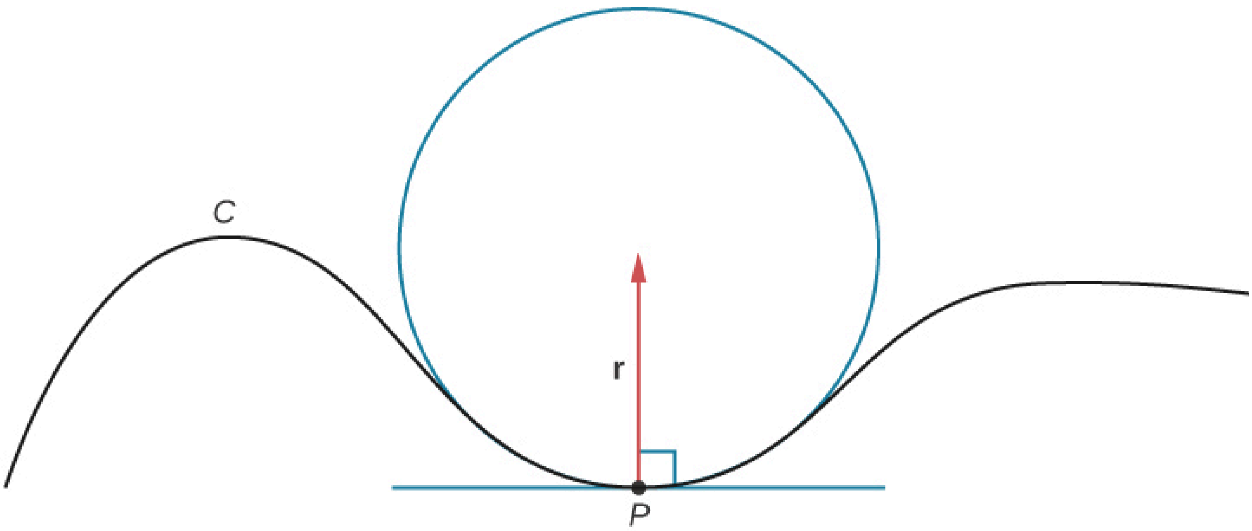
2. At each t_i, define $\mathbf{r}_i = \mathbf{r}(t_i)$.

3. The polygonal length is

$$S_n = \sum_{i=1}^{n} \left|\left|\mathbf{r}_i - \mathbf{r}_{i-1}\right|\right|$$

4. When $\mathbf{r}(t)$ has a **continuous derivative** $\mathbf{v}(t) = \mathbf{r}'(t)$, the arc length is given by the integral

$$S = \lim_{n\to\infty} S_n = \int_a^b \left|\mathbf{r}'(t)\right| dt = \int_a^b \left|\mathbf{v}(t)\right| dt$$



Example: Circular Helix

A **circular helix** in $\mathbb{R}^3$ can be parametrized by

$$\mathbf{r}(t) = \langle a\cos t,\ a\sin t,\ b\,t\rangle, \quad 0 \le t \le T$$

1. Compute $\mathbf{r}'(t)$:

$$\mathbf{r}'(t) = \big\langle -a\sin t,\ a\cos t,\ b\big\rangle$$

2. Find its magnitude:

$$\left|\mathbf{r}'(t)\right| = \sqrt{(-a\sin t)^2 + (a\cos t)^2 + b^2} = \sqrt{a^2(\sin^2 t + \cos^2 t) + b^2} = \sqrt{a^2 + b^2}$$

3. The arc length from $t = 0$ to $t = 2\pi$ is

$$S = \int_0^{2\pi} \sqrt{a^2 + b^2}\, dt = \sqrt{a^2 + b^2}\,(2\pi - 0) = 2\pi\sqrt{a^2 + b^2}$$

4. **Remark:** The arc length is independent of how we choose to parametrize the curve. If there are multiple valid parametrizations, they all yield the same length.

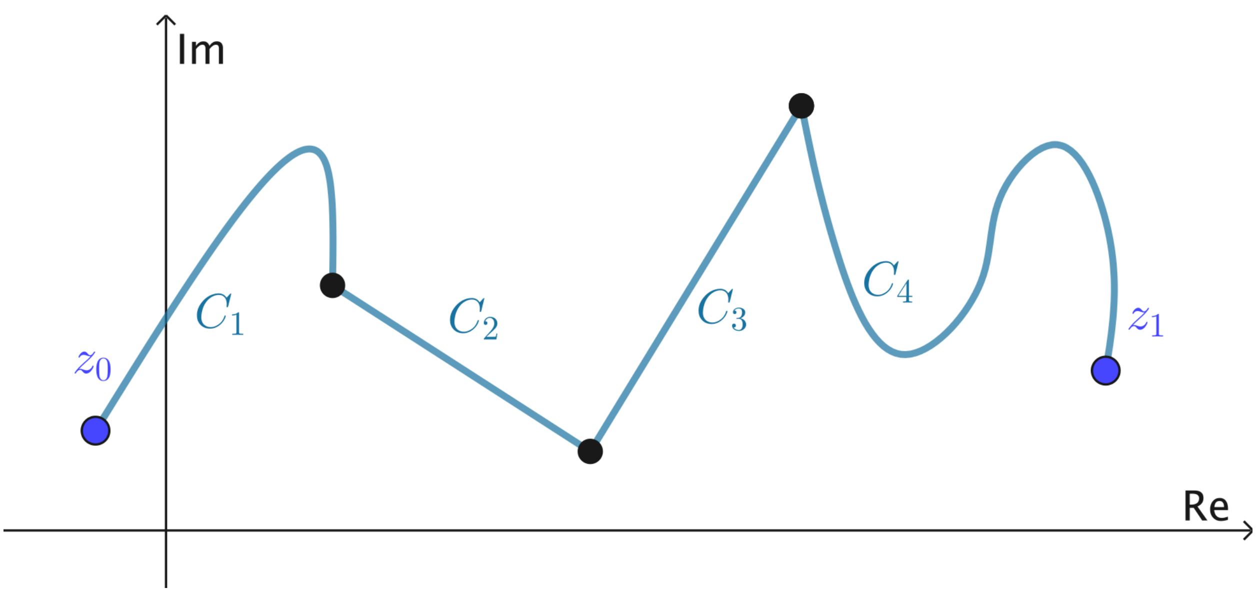
Piecewise Smooth Curves

A **piecewise smooth curve** C (of order 1) is formed by a **finite number** of smooth arcs. We can write

$$C = C_1 + C_2 + \cdots + C_k$$

where each C_i is smooth on its own. Then, the total length is the sum of the lengths of these arcs:

$$S(C) = \sum_{i=1}^{k} S(C_i).$$



Condition for Constant Speed

Although slightly outside the main topic, it is often shown that a particle moves with **constant speed** if and only if its **acceleration $\mathbf{a}(t)$** is **perpendicular** (orthogonal) to its **velocity $\mathbf{v}(t)$** for all t in the interval of motion. Formally,

> If $\mathbf{v}(t) \cdot \mathbf{a}(t) = 0$ for all t, then $\left|\mathbf{v}(t)\right|$ remains constant.

Sketch of Proof:

- Since $\frac{d}{dt}\left(\frac{1}{2}|\mathbf{v}(t)|^2\right) = \mathbf{v}(t) \cdot \mathbf{a}(t)$, if this derivative is zero at all t, then $\frac{1}{2}|\mathbf{v}(t)|^2$ is constant. Hence, $\left|\mathbf{v}(t)\right|$ is also constant.

Final Summary & Takeaways

- **Parametrization** allows us to describe curves in $\mathbb{R}^2$ or $\mathbb{R}^3$ via vector-valued functions $\mathbf{r}(t)$.
- **Closed curves** satisfy $\mathbf{r}(a) = \mathbf{r}(b)$, and we say a curve is **not-self-intersecting** if $\mathbf{r}(t)$ is injective except for endpoints (if closed).
- **Smoothness** refers to the continuity of derivatives. A curve is **smooth of order 1** if $\mathbf{r}'(t)$ is continuous.
- **Arc length** is computed via the integral $\int_a^b |\mathbf{r}'(t)|\, dt$.
- **Piecewise smooth curves** are composed of finitely many smooth arcs, and their lengths add up.
- **Constant speed** occurs precisely when $\mathbf{a}(t)$ is perpendicular to $\mathbf{v}(t)$.

All of these concepts set the stage for deeper explorations in vector calculus and the geometry of curves in multiple dimensions.

Raw Notes

Raw Notes

6. Frenet–Serret Frame, Curvature, and Torsion

Prerequisites

Assumes a regular, sufficiently smooth space curve \Assumes a regular, sufficiently smooth space curve $\mathbf{r}(t)$ with nonzero velocity $\mathbf{r}'(t)$. We will often reparameterize by arc length s, so that $\|\frac{d\mathbf{r}}{ds}\| = 1.$ $\mathbf{r}(t)$ with nonzero velocity $\mathbf{r}'(t)$. We will often reparameterize by arc length s, so that $\|\frac{d\mathbf{r}}{ds}\| = 1.$

Unit Tangent Vector

> **Unit Tangent Vector ($\mathbf{T}$)**
>
> The unit tangent vector to the curve is
>
> $$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|},$$
>
> or, when using arc length s, $\mathbf{T}(s) = \frac{d\mathbf{r}}{ds}$.

Curvature and Unit Normal

Curvature (κ)

A scalar measure of how sharply the curve bends, defined by

$$\kappa(s) \;=\; \left\| \frac{d\mathbf{T}}{ds} \right\|.$$

Equivalently, in any parameter t,

$$\kappa \;=\; \frac{\|\mathbf{r}' \times \mathbf{r}''\|}{\|\mathbf{r}'\|^3}.$$

Unit Normal Vector ($\mathbf{N}$)

The principal normal points toward the center of curvature:

$$\mathbf{N}(s) \;=\; \frac{d\mathbf{T}/ds}{\|d\mathbf{T}/ds\|}\,.$$

Binormal and Torsion

Binormal Vector ($\mathbf{B}$)

Defined as the cross product of $\mathbf{T}$ and $\mathbf{N}$:

$$\mathbf{B}(s) \;=\; \mathbf{T}(s) \times \mathbf{N}(s).$$

Torsion (τ)

Measures how the curve twists out of the osculating plane:

$$\tau(s) \;=\; -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N}(s).$$

In a general parameter t,

$$\tau \;=\; \frac{(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''}{\|\mathbf{r}' \times \mathbf{r}''\|^2}.$$

Frenet–Serret Formulas

Frenet–Serret Formulas

For a unit-speed curve parameterized by arc length s, the derivatives of the frame $(\mathbf{T}, \mathbf{N}, \mathbf{B})$ are:

$$\frac{d\mathbf{T}}{ds} = \kappa\,\mathbf{N},$$
$$\frac{d\mathbf{N}}{ds} = -\,\kappa\,\mathbf{T} \;+\; \tau\,\mathbf{B},$$
$$\frac{d\mathbf{B}}{ds} = -\,\tau\,\mathbf{N}.$$

These equations describe the instantaneous rotation of the orthonormal triad along the curve.

7. Fundamental Theorem of Space Curves & Curvature, Torsion for General Parameterization

Objective & Scope

This note states the **Fundamental Theorem of Space Curves**, which asserts existence and uniqueness of a space curve given curvature and torsion, and then summarizes the formulas for **curvature** and **torsion** when the curve is given by a general parameter t.

Curvature and Torsion for General Parameterization

> **Curve Parameterization:** A smooth vector-valued function $\mathbf{r}(t)$, $t \in I$, with $\mathbf{r}'(t) \neq \mathbf{0}$ for all t.

Curvature

> **Curvature (κ) for general t:** Measures how rapidly the curve deviates from a straight line, given by

> $$\kappa(t) \;=\; \frac{\bigl\|\mathbf{r}'(t) \times \mathbf{r}''(t)\bigr\|}{\|\mathbf{r}'(t)\|^3}\,.$$

- **Interpretation:**
  A larger κ indicates tighter bending of the curve at that point.

Torsion

> **Torsion (τ) for general t:** Measures how rapidly the curve departs from its osculating plane, given by
>
> $$\tau(t) \;=\; \frac{\bigl(\mathbf{r}'(t) \times \mathbf{r}''(t)\bigr) \;\cdot\; \mathbf{r}'''(t)}{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|^2}\,.$$

- **Interpretation:**
  A nonzero τ indicates twisting of the curve out of the plane of curvature.

Fundamental Theorem of Space Curves

> **Fundamental Theorem of Space Curves:** Let $\kappa(s)$ and $\tau(s)$ be two smooth functions defined on an interval, with $\kappa(s) > 0$. Then there exists a regular, unit-speed space curve $\mathbf{r}(s)$, unique up to a rigid motion (rotation and translation), whose curvature and torsion are exactly $\kappa(s)$ and $\tau(s)$, respectively.

- **Existence:**

  One can integrate the Frenet–Serret system with given $\kappa(s)$ and $\tau(s)$ to recover $\mathbf{T}(s)$, $\mathbf{N}(s)$, and $\mathbf{B}(s)$, and thereby reconstruct $\mathbf{r}(s)$.

- **Uniqueness (up to Rigid Motion):**

  Any two curves with the same prescribed $\kappa(s)$ and $\tau(s)$ differ only by a fixed orthogonal transformation and translation in space.

Summary

- **General-parameter formulas** allow computation of curvature and torsion directly from $\mathbf{r}(t)$.

- The **Fundamental Theorem** guarantees that curvature and torsion completely determine the shape of a space curve (modulo rigid motions), encapsulating the intrinsic geometry of the curve.

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8. Partial Differentiation & Functions of Several Variables

Functions of Several Variables

> **Multivariable Function:** A mapping $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$ (or($\mathbb{R}^m$) that assigns to each point $(x_1, x_2, \dots, x_n) \in D$ a single real value (or an m-vector).
>
> - **Domain** D is a subset of $\mathbb{R}^n$.
> - **Codomain** is typically $\mathbb{R}$ for scalar-valued functions.

> **Level Sets & Contours:** For a scalar $f(x, y)$, the set $\{(x, y) \mid f(x, y) = c\}$ is a **level curve** (contour) in the plane. In higher dimensions, $\{\,\mathbf{x} \mid f(\mathbf{x}) = c\}$ is a **level surface**.

Partial Derivatives

> **Partial Derivative (First Order):** The rate of change of f with respect to one variable x_i, holding all other variables constant:
>
> $$\frac{\partial f}{\partial x_i}(\mathbf{x}) = \lim_{h \to 0} \frac{f(x_1, \dots, x_i + h, \dots, x_n) - f(x_1, \dots, x_i, \dots, x_n)}{h}.$$

> Notations include f_{x_i}, $\partial_i f$, or $D_{x_i} f$.

> **Existence:** A function may have partial derivatives without being continuous or jointly differentiable.

Differentiability and the Total Differential

> **Differentiability:** f is differentiable at $\mathbf{a}$ if there exists a linear map (the **total derivative**) $Df(\mathbf{a})$ such that
>
> $$f(\mathbf{a}+\mathbf{h}) = f(\mathbf{a}) + Df(\mathbf{a})[\mathbf{h}] + o(\|\mathbf{h}\|),$$
>
> where $o(\|\mathbf{h}\|)/\|\mathbf{h}\| \to 0$ as $\|\mathbf{h}\| \to 0$.

> **Total Differential:** For a differentiable scalar function $f(x, y)$,
>
> $$df = f_x\,dx + f_y\,dy,$$
>
> where f_x and f_y are partial derivatives.

Summary

- **Partial derivatives** capture change in one coordinate direction.
- **Differentiability** ensures a good linear approximation (total differential).

---

9. Limits and Continuity

> **LIMIT (Multivariable):** For a function $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$ and a point $\mathbf{a} \in D$,
>
> $$\lim_{\mathbf{x} \to \mathbf{a}} f(\mathbf{x}) = L$$
>
> means that for every $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $0 < \|\mathbf{x} - \mathbf{a}\| < \delta$, then $|f(\mathbf{x}) - L| < \varepsilon$.

> **CONTINUITY (Multivariable):** f is continuous at $\mathbf{a}$ if
>
> $$\lim_{\mathbf{x} \to \mathbf{a}} f(\mathbf{x}) = f(\mathbf{a})$$
>
> Continuity on a domain means this holds at every point in the domain. Continuity implies that small changes in each coordinate yield arbitrarily small changes in the function's value.

Partial Derivatives

> **PARTIAL DERIVATIVE:** The partial derivative of f with respect to its i-th variable at $\mathbf{x}$ is

$$\frac{\partial f}{\partial x_i}(\mathbf{x}) = \lim_{h \to 0} \frac{f(x_1, \ldots, x_i + h, \ldots, x_n) \; - \; f(x_1, \ldots, x_i, \ldots, x_n)}{h},$$

provided this limit exists.

NOTATION: Common notations include f_{x_i}, $\partial_{x_i} f$, or $D_{x_i} f$.

EXISTENCE & PROPERTIES:

- Partial derivatives may exist individually without guaranteeing overall differentiability or continuity of f.
- If all first-order partials exist and are continuous on a domain, then f is differentiable there, and its total differential is given by

$$df = \sum_{i=1}^{n} f_{x_i} \, dx_i.$$

10. Tangent Planes, Higher Order Derivatives

Tangent Planes and Normal Lines

> **Surface in Explicit Form:** A surface given by $z = f(x, y)$ with continuous first partials has at each point $\big(x_0, y_0, f(x_0, y_0)\big)$ a unique tangent plane.

> **Tangent Plane (Explicit):**
>
> $$z - z_0 \;=\; f_x(x_0, y_0)\,(x - x_0) \;+\; f_y(x_0, y_0)\,(y - y_0),$$
>
> where f_x and f_y are the first partial derivatives.

> **Surface in Implicit Form:** A surface defined by $F(x, y, z) = 0$ with $\nabla F \neq \mathbf{0}$ at a point.

> **Tangent Plane (Implicit):**
>
> $$F_x(x_0, y_0, z_0)\,(x - x_0) \;+\; F_y(x_0, y_0, z_0)\,(y - y_0) \;+\; F_z(x_0, y_0, z_0)\,(z - z_0) \;=\; 0.$$

> **Normal Line:** The line through $\big(x_0, y_0, z_0\big)$ in the direction of the surface normal $\mathbf{n} = \nabla F(x_0, y_0, z_0)$.
>
> Parametric form:

> $$(x, y, z) = (x_0, y_0, z_0) + t\,(F_x, F_y, F_z).$$

Higher Order Derivatives

> **Second and Higher-Order Partial Derivatives:** For $f\colon \mathbb{R}^n \to \mathbb{R}$, the second partials are
>
> $$f_{x_i x_j} = \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right)$$
>
> Higher-order derivatives are defined by iterating this process.

> **Mixed Partial Symmetry (Clairaut's Theorem):** If the mixed partials $f_{x_i x_j}$ and $f_{x_j x_i}$ are continuous in a neighborhood, then
>
> $$f_{x_i x_j} \;=\; f_{x_j x_i}.$$

> **Multi-Index Notation:** For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$,
>
> $$D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1}\,\partial x_2^{\alpha_2}\,\cdots\,\partial x_n^{\alpha_n}},$$
>
> where $|\alpha| = \sum \alpha_i$.

> **Hessian Matrix:** The matrix of second partials for $f(x, y)$:
>
> $$H_f(x, y) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}.$$
>
> It encodes curvature information and is symmetric if mixed partials commute.

> **Higher-Order Differentials:** The total differential extends to higher order via Taylor's theorem, using the derivatives up to the desired order.

11. Chain Rule, Linear Approximation, Differentiability, Differentials

Chain Rule

> **Chain Rule (Multivariable):** If $F : \mathbb{R}^m \to \mathbb{R}$ is differentiable at $\mathbf{y} = g(\mathbf{x})$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $\mathbf{x}$, then the composite $F \circ g$ is differentiable at $\mathbf{x}$ and
>
> $$D(F \circ g)(\mathbf{x}) = DF\big(g(\mathbf{x})\big) \cdot Dg(\mathbf{x})\,.$$
>
> Equivalently, for scalar functions $z = F(u, v, \dots)$ and each intermediate variable $u = u(x, y, \dots)$,
>
> $$\frac{\partial z}{\partial x} = \sum_i \frac{\partial F}{\partial u_i}\,\frac{\partial u_i}{\partial x}\,,$$
>
> and similarly for other variables.

Linear Approximation

> Linear Approximation (Tangent Plane Approximation):
>
> For a differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ at a point $\mathbf{a}$, the linear approximation (or first-order Taylor expansion) is
>
> $$f(\mathbf{a}+\mathbf{h}) \approx f(\mathbf{a}) + Df(\mathbf{a})[\mathbf{h}] = f(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot \mathbf{h}\,.$$
>
> where $\mathbf{h}$ is a small increment vector and ∇f is the gradient.

Differentiability

> Differentiability:
>
> A function $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\mathbf{a}$ if there exists a linear map L (the total derivative) such that
>
> $$\lim_{\|\mathbf{h}\|\to 0} \frac{f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - Df(\mathbf{a})[\mathbf{h}]}{\|\mathbf{h}\|} = 0.$$
>
> In that case, $L = Df(\mathbf{a})$ and f admits the linear approximation above. Differentiability implies continuity and the existence of all partial derivatives, but the converse requires those partials to be continuous.

Differentials

> Differential (df):
>
> The differential df of a differentiable function $f(x_1, \ldots, x_n)$ at $\mathbf{a}$ is the linear form
>
> $$df(\mathbf{a}) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\mathbf{a})\, dx_i,$$
>
> where dx_i denotes an infinitesimal change in x_i. The differential gives the best linear estimate of the change in f corresponding to small changes $(dx_1, \ldots, dx_n)$.

---

12. Linearization, Differentiability

Linearization

> Linearization:
>
> The process of approximating a differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ near a point $\mathbf{a}$ by its first-order Taylor polynomial (the tangent hyperplane).
>
> The linearization L at $\mathbf{a}$ is the affine map
>
> $$L(\mathbf{x}) \;=\; f(\mathbf{a}) \;+\; Df(\mathbf{a})[\mathbf{x}-\mathbf{a}] \;=\; f(\mathbf{a}) \;+\; \nabla f(\mathbf{a}) \cdot (\mathbf{x}-\mathbf{a}),$$
>
> which provides the best linear approximation of f for $\mathbf{x}$ close to $\mathbf{a}$.

Differentiability

> Differentiability:
>
> A function $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at a point $\mathbf{a}$ if there exists a linear map $Df(\mathbf{a})$ (the total derivative) such that
>
> $$\lim_{\mathbf{h}\to\mathbf{0}} \frac{f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - Df(\mathbf{a})[\mathbf{h}]}{\|\mathbf{h}\|} = 0.$$

> Differentiability implies continuity at $\mathbf{a}$ and guarantees that f can be locally approximated by its linearization.

---

13. Gradients and Directional Derivatives

Gradients and Directional Derivatives

> **Gradient (∇f):** For a differentiable scalar function $f(x_1, x_2, \dots, x_n)$, the gradient is the vector of its first partial derivatives:
>
> $$\nabla f(\mathbf{x}) = \big(f_{x_1}(\mathbf{x}), f_{x_2}(\mathbf{x}), \dots, f_{x_n}(\mathbf{x})\big).$$
>
> It points in the direction of steepest increase of f and its magnitude is the maximum rate of change.

> **Directional Derivative ($D_{\mathbf{u}}f$):** The rate of change of f at $\mathbf{x}$ in the direction of a unit vector $\mathbf{u}$ is given by
>
> $$D_{\mathbf{u}}f(\mathbf{x}) = \lim_{h \to 0} \frac{f(\mathbf{x} + h\,\mathbf{u}) - f(\mathbf{x})}{h} = \nabla f(\mathbf{x}) \cdot \mathbf{u}.$$
>
> This scalar measures how f changes per unit displacement along $\mathbf{u}$.

Gradient in 3D

> **Gradient in Three Dimensions:** For a function $f(x, y, z)$, the gradient is
>
> $$\nabla f(x, y, z) = \big(f_x(x, y, z),\ f_y(x, y, z),\ f_z(x, y, z)\big).$$
>
> - **Interpretation:** At each point, ∇f is orthogonal to the level surface $f(x, y, z) = \text{constant}.$
> - **Properties:**
>   - Its direction is that of maximal increase of f.
>   - Its magnitude $\|\nabla f\|$ equals the maximal directional derivative at that point.
>   - It serves as the normal vector in the tangent-plane equation for the surface $f(x, y, z) = c.$

14. Midterm Preparation 1

> **Curve Parameterization:** $\mathbf{r}(t)$ with $\mathbf{r}'(t) \neq \mathbf{0}$. When reparameterized by arc length s, $\|\frac{d\mathbf{r}}{ds}\| = 1$.

Unit Tangent Vector

- **General t:**

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}.$$

- **Arc length s:**

$$\mathbf{T}(s) = \frac{d\mathbf{r}}{ds}.$$

Curvature

- **Arc length s:**

$$\kappa(s) = \left\| \frac{d\mathbf{T}}{ds} \right\|.$$

- **General t:**

$$\kappa(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}.$$

Principal Normal Vector

$$\mathbf{N}(s) = \frac{\frac{d\mathbf{T}}{ds}}{\|\frac{d\mathbf{T}}{ds}\|}.$$

Binormal Vector

$$\mathbf{B}(s) = \mathbf{T}(s) \times \mathbf{N}(s).$$

Torsion

- **Arc length s:**

$$\tau(s) = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N}(s).$$

- **General t:**

$$\tau(t) = \frac{\big(\mathbf{r}'(t) \times \mathbf{r}''(t)\big) \cdot \mathbf{r}'''(t)}{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|^2}.$$

Frenet–Serret Formulas (w.r.t. s)

$$\begin{aligned}
\frac{d\mathbf{T}}{ds} &= \kappa\,\mathbf{N}, \\
\frac{d\mathbf{N}}{ds} &= -\kappa\,\mathbf{T} + \tau\,\mathbf{B}, \\
\frac{d\mathbf{B}}{ds} &= -\tau\,\mathbf{N}.
\end{aligned}$$

---

15. Midterm Preparation 2

Limits and Continuity

$$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = L \quad \Longleftrightarrow \quad \forall\,\varepsilon > 0,\ \exists\,\delta > 0 :\ 0 < \|\mathbf{x}-\mathbf{a}\| < \delta \implies |f(\mathbf{x}) - L| < \varepsilon$$

$$f \text{ continuous at } \mathbf{a} \iff \lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = f(\mathbf{a})$$

Partial Derivatives

$$\frac{\partial f}{\partial x_i}(\mathbf{x}) = \lim_{h\to 0} \frac{f(x_1,\ldots,x_i+h,\ldots,x_n) - f(x_1,\ldots,x_i,\ldots,x_n)}{h}$$

Differentiability & Total Differential

$$f(\mathbf{a}+\mathbf{h}) = f(\mathbf{a})Df(\mathbf{a})[\mathbf{h}]o(\|\mathbf{h}\|) \quad , \quad Df(\mathbf{a})[\mathbf{h}] = \nabla f(\mathbf{a}) \cdot \mathbf{h}$$

$$df = \sum_{i=1}^{n} f_{x_i}\, dx_i$$

Linear Approximation & Linearization

$$L(\mathbf{x}) = f(\mathbf{a})\nabla f(\mathbf{a}) \cdot \big(\mathbf{x} - \mathbf{a}\big)$$

Chain Rule

If $\mathbf{y} = g(\mathbf{x})$ and F differentiable at $\mathbf{y}$, then

$$D(F \circ g)(\mathbf{x}) = DF\big(g(\mathbf{x})\big) \,\cdot\, Dg(\mathbf{x}) \quad \Longleftrightarrow \quad \frac{\partial}{\partial x_j} F(g(\mathbf{x})) = \sum_i \frac{\partial F}{\partial y_i} \frac{\partial g_i}{\partial x_j}$$

Gradient & Directional Derivative

$$\nabla f(\mathbf{x}) = \big(f_{x_1}, f_{x_2}, \ldots, f_{x_n}\big) \quad , \quad D_{\mathbf{u}} f(\mathbf{x}) = \nabla f(\mathbf{x}) \cdot \mathbf{u}$$

Tangent Plane & Normal Line

Explicit: $z = f(x, y)$

$$z - z_0 = f_x(x_0, y_0)\,(x - x_0) f_y(x_0, y_0)\,(y - y_0)$$

Implicit: $F(x, y, z) = 0$

$$F_x(x_0, y_0, z_0)\,(x - x_0) + F_y(x_0, y_0, z_0)\,(y - y_0) + F_z(x_0, y_0, z_0)\,(z - z_0) = 0$$

Higher-Order Derivatives & Clairaut's Theorem

$$f_{x_i x_j} = \frac{\partial}{\partial x_j}\Big(f_{x_i}\Big) \quad , \quad f_{x_i x_j} = f_{x_j x_i} \text{ if mixed partials are continuous}$$

---

16. Lagrange Multipliers

Theorem

Our aim is to maximize or minimize $f(x, y)$ subject to $\rho(x, y) = 0$.

Theorem: Suppose that f and ρ have continuous first partial derivatives near $P_0 = (x_0, y_0)$ on the curve C with equation $\rho(x, y) = 0$. Suppose also that, when restricted to points on C, $f(x, y)$ has a local max or min at P_0. Finally, suppose that

- (i) P_0 is not an endpoint of C, and
- (ii) $\nabla\rho(P_0) \neq 0$.

Then, there exists a number λ_0 such that

$$\nabla L(x_0, y_0, \lambda_0) = 0 \quad \text{where} \quad L(x, y, \lambda) = f(x, y) + \lambda\rho(x, y).$$

This leads to the following system of equations:

$$\begin{aligned} f_1(x_0, y_0) + \lambda_0\rho_1(x_0, y_0) &= 0 \\ f_2(x_0, y_0) + \lambda_0\rho_2(x_0, y_0) &= 0 \\ \rho(x_0, y_0) &= 0 \end{aligned}$$

which is equivalent to

$$f_1(x_0, y_0)\rho_2(x_0, y_0) = f_2(x_0, y_0)\rho_1(x_0, y_0)$$
$$\rho(x_0, y_0) = 0$$

Example 1: Shortest Distance from the Origin to the Curve $x^2y = 16$

We want to minimize $f(x, y) = x^2 + y^2$ subject to $\rho(x, y) = x^2y - 16$.

$$f_1(x, y) = 2x, \quad f_2(x, y) = 2y, \quad \rho_1(x, y) = 2xy, \quad \rho_2(x, y) = x^2.$$

The equations are:

$$2x + \lambda_0 \cdot 2xy = 0$$
$$2y + \lambda_0 \cdot x^2 = 0$$
$$x^2y = 16$$

Solving these, we find:

- $x = 0$ or $x^2 = 2y^2$,
  - $x = 0$ or $x = \pm y\sqrt{2}$.

Substituting back into $x^2y = 16$:

- $2y^3 = 16 \quad \Rightarrow \quad y = 2$.

Thus, at $(2\sqrt{2}, 2)$ and $(-2\sqrt{2}, 2)$, the distance is minimum. It is:

$$\sqrt{8 + 4} = \sqrt{12}.$$

(This cannot be a maximum because, for example, the distance from $(1, 1)$ to the origin is greater.)

Example 2: Find the Closest and Farthest Points from the Origin

Find the points on the curve $3x^2 + 12xy + 8y^2 = 100$ that are closest and farthest away from the origin.

We want to maximize and minimize $f(x, y) = x^2 + y^2$ subject to $\rho(x, y) = 3x^2 + 12xy + 8y^2 - 100$.

$$L(x, y, \lambda) = f(x, y) + \lambda \cdot \rho(x, y).$$

The equations are:

$$\begin{aligned} 0 &= \frac{\partial L}{\partial x} = 2x + \lambda(6x + 12y) \\ 0 &= \frac{\partial L}{\partial y} = 2y + \lambda(12x + 16y) \\ 0 &= \frac{\partial L}{\partial \lambda} = 3x^2 + 12xy + 8y^2 - 100 \end{aligned}$$

Solving the system of equations:

$$2x(12x + 16y) = 2y(6x + 12y).$$

This leads to the following system and solutions for x and y:

For $x = 2$, we get $y = 1$ or $y = -4$.

For $x = -2$, we get $y = 1$ or $y = 4$.

Thus, the candidate points are $(2, 1)$, $(2, -4)$, $(-2, 1)$, $(-2, 4)$.

The function values for these points are:

$$f(2, 1) = 5, \quad f(2, -4) = 20, \quad f(-2, 1) = 5, \quad f(-2, -4) = 20.$$

Therefore, $(2, 1)$ and $(-2, 1)$ are closest to the origin, and $(2, -4)$ and $(-2, -4)$ are farthest away from the origin.

---

17. Double Integrals & Iterated Integrals

Double Integrals

Definitions

> **DOMAIN D:** A region in the xy-plane over which a function $f(x, y)$ is integrated.
>
> **DOUBLE INTEGRAL:**
>
> $$\iint_D f(x, y)\, dA$$
>
> represents the "limit" of Riemann sums and gives the volume under $z = f(x, y)$ above D.
>
> **ELEMENT OF AREA:**
>
> $$dA = dx\, dy = dy\, dx$$

Riemann Sum & Integrability

1. Partition a rectangle $D = [a, b] \times [c, d]$ into subrectangles R_{ij} with sides Δx_i, Δy_j.
2. Choose sample point (x_{ij}, y_{ij}) *in each* R_{ij}.

3. **Riemann sum:**

$$R(f,P) \;=\; \sum_{i=1}^{m}\sum_{j=1}^{n} f\bigl(x_{ij}^*,\, y_{ij}^*\bigr)\,\Delta x_i\,\Delta y_j.$$

4. **Integrability:** f is integrable if there exists I such that for every $\varepsilon > 0$ a partition norm $\|P\|$ small enough implies $\bigl|R(f,P) - I\bigr| < \varepsilon$.

Double Integral over General Domain

> **EXTENDED FUNCTION:**
>
> $$\hat{f}(x,y) = \begin{cases} f(x,y), & (x,y) \in D, \\ 0, & \text{otherwise.} \end{cases}$$
>
> Then $\iint_D f\,dA = \iint_R \hat{f}\,dA$ for any rectangle $R \supset D$.

Theorems & Properties

> **CONTINUITY ⇒ INTEGRABILITY:** If f is continuous on a closed, bounded D with piecewise-smooth boundary, then f is integrable.
>
> **LINEARITY:**
>
> $$\iint_D \bigl[L\,f + M\,g\bigr]\,dA = L\iint_D f\,dA + M\iint_D g\,dA.$$
>
> **ORDER:** If $f \le g$ on D, then
>
> $$\iint_D f\,dA \le \iint_D g\,dA.$$
>
> **TRIANGLE INEQUALITY:**
>
> $$\Bigl|\iint_D f\,dA\Bigr| \le \iint_D |f|\,dA.$$
>
> **ADDITIVITY:** If $D = D_1 \cup D_2$ with nonoverlapping D_i,

> $$\iint_D f\,dA = \iint_{D_1} f\,dA + \iint_{D_2} f\,dA.$$
>
> **SYMMETRY (Odd Functions):**
>
> If D is symmetric about the y-axis and f is odd in x, then $\iint_D f\,dA = 0$.
>
> Similarly for odd in y over x-axis symmetry.

Examples

Example 1: Approximate $\displaystyle\iint_{[0,1]^2} (x^2 + y)\,dA$ using 4 subsquares and centers.

- Centers: $(\frac{1}{4}, \frac{1}{4}), (\frac{3}{4}, \frac{1}{4}), (\frac{1}{4}, \frac{3}{4}), (\frac{3}{4}, \frac{3}{4})$.
- Each $\Delta A = \frac{1}{4}$.

$$R \approx \sum f(x^{,}y^{)}\,\Delta A = \left(\tfrac{1}{16} + \tfrac{1}{4}\right)\tfrac{1}{4} + \cdots = \tfrac{3}{16}.$$

Example 2: Volume under $z = f(x, y) = \sqrt{1 - x^2 - y^2}$ over $x^2 + y^2 \leq 1$.

- Recognize hemisphere of radius 1.

$$\iint_{x^2+y^2\leq 1} \sqrt{1 - x^2 - y^2}\,dA = \tfrac{2\pi}{3}.$$

Iteration of Double Integrals in Cartesian Coordinates

Domain Types

> **y-SIMPLE Domain:** Bounded by vertical lines $x = a$, $x = b$ and curves $y = c(x)$, $y = d(x)$.
>
> **x-SIMPLE Domain:** Bounded by horizontal lines $y = c$, $y = d$ and curves $x = a(y)$, $x = b(y)$.
>
> **REGULAR Domain:** Finite union of nonoverlapping simple domains.

Fubini's Theorem for Simple Domains

> **THEOREM:** If f is continuous on a bounded y-simple domain D with $a \leq x \leq b$, $c(x) \leq y \leq d(x)$, then

> $$\iint_D f(x,y)\,dA = \int_{x=a}^{b}\int_{y=c(x)}^{d(x)} f(x,y)\,dy\,dx.$$
>
> Similarly for x-simple domains with $dx\,dy$ order.

Notation

$$\iint_D f\,dA = \int\int f(x,y)\,dx\,dy = \int_a^b\int_{c(x)}^{d(x)} f\,dy\,dx \quad \text{etc.}$$

Examples

Example 3: Volume over square $Q:\ 0 \le x \le 1,\ 1 \le y \le 2$ under plane $z = 4 - x - y$.

$$V = \int_{y=1}^{2}\int_{x=0}^{1}(4-x-y)\,dx\,dy = \int_1^2\left[4x - \tfrac{x^2}{2} - xy\right]_0^1 dy = \int_1^2\left(\tfrac{7}{2} - y\right)dy = 2.$$

Example 4: $\iint_T xy\,dA$ over triangle T with vertices $(0,0),(1,0),(1,1)$.

- Describe as $0 \le x \le 1,\ 0 \le y \le x$.

$$\int_{x=0}^{1}\int_{y=0}^{x} xy\,dy\,dx = \int_0^1 x\left[\tfrac{y^2}{2}\right]_0^x dx = \int_0^1 \tfrac{x^3}{2}\,dx = \tfrac{1}{8}.$$

Example 5: $\displaystyle\int_{x=0}^{1}\int_{y=1}^{\sqrt{x}} e^{y^3}\,dy\,dx.$

- (Set up iterated integral; evaluation may require numerical methods or change of order.)

Final Summary & Takeaways

- **Double integral** gives volume under $z = f(x,y)$.
- **Riemann sum** definition and **integrability** criterion.
- **Properties:** linearity, order, additivity, symmetry for odd functions.
- **Fubini's Theorem**: evaluate $\iint_D$ as iterated $\int\int$.
- **Common mistake:** forgetting to adjust limits when changing integration order.

18. Improper Integrals & Coordinate Transformations

Improper Double Integrals

Definitions

> **Improper Domain:** A double integral $\iint_D f(x, y)\, dA$ is improper if the region D is unbounded.
>
> **Unbounded Integrand:** The integral is also improper if $f(x, y)$ becomes unbounded on or near D or its boundary.

Convergence Criterion

> **Nonnegative Functions:** If $f(x, y) \geq 0$ on D, then $\iint_D f\, dA$ either converges to a finite value or diverges to $+\infty$.
>
> **Comparison Tests:** Analogous to single-variable tests (e.g., p-integral test).

Example: Exponential Integrand over an Unbounded Region

Problem: Evaluate

$$
\iint_R e^{-x^2}\,dA, \quad R = \{(x, y) \mid x \geq 0,\ -x \leq y \leq x\}.
$$

Solution:

Express as an iterated integral:

$$
\int_0^\infty \Big(\int_{-x}^{x} e^{-x^2}\,dy\Big)\,dx.
$$

Inner integral gives $2x\,e^{-x^2}$.

Then

$$
\int_0^\infty 2x\,e^{-x^2}\,dx = \left[-e^{-x^2}\right]_0^\infty = 1.
$$

Example: Singularity near the Boundary

Problem: Evaluate

$$
\iint_D \frac{1}{(x+y)^2}\,dA, \quad D = \{(x, y) \mid 0 \leq x \leq 1,\ 0 \leq y \leq x^2\}.
$$

Solution:

Write as a limit to handle the singularity at $x = 0$:

$$
\lim_{c \to 0^+} \int_c^1 \int_0^{x^2} \frac{1}{(x+y)^2}\,dy\,dx.
$$

Inner antiderivative: $-\dfrac{1}{x+y}\Big|_0^{x^2} = \dfrac{1}{x} - \dfrac{1}{x(1+x)}$.

Simplify and integrate to get $\ln 2$.

Absolute Convergence

> **Absolute Convergence Criterion:** If $\iint_D |f(x, y)|\,dA$ converges, then the original integral $\iint_D f(x, y)\,dA$ also converges.

Mean-Value Theorem for Double Integrals

> **Theorem:** For a continuous function f on a closed, bounded, connected domain D of area A, there exists $(x_0, y_0) \in D$ such that
>
> $$\iint_D f(x,y)\,dA = A\,f(x_0,y_0).$$
>
> **Average Value:**
>
> $$\overline{f} = \frac{1}{A}\iint_D f(x,y)\,dA.$$

Example: Average of a Quadratic Function

Compute the average of $f(x,y) = x^2 + y^2$ over the triangle with vertices $(0,0)$, $(1,0)$, and $(1,1)$.

The area is $A = \frac{1}{2}$. Set up $\dfrac{1}{A}\displaystyle\iint_T (x^2+y^2)\,dA$ and evaluate.

Polar Coordinates & Change of Variables

Polar Coordinates Review

> **Coordinate Transformation:** $x = r\cos\theta,\ y = r\sin\theta,\ r \geq 0,\ 0 \leq \theta < 2\pi.$
>
> **Area Element:** $dA = r\,dr\,d\theta.$

Double Integrals in Polar Form

> **Fubini's Theorem (Polar):** If D is given by $\alpha \leq \theta \leq \beta,\ 0 \leq r \leq R(\theta)$, then
> $$\iint_D f(x,y)\,dA = \int_\alpha^\beta \int_0^{R(\theta)} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta.$$

Example: Volume under a Paraboloid

Evaluate $\displaystyle\iint_{x^2+y^2\leq 1} (1 - x^2 - y^2)\,dA$ by using polar coordinates.

Substitute $x^2 + y^2 = r^2$ and integrate from $r = 0$ to 1, $\theta = 0$ to 2π, yielding $\pi/2$.

Example: Annular Sector Integral

Evaluate $\iint_R \frac{y^2}{x^2}\,dA$ where R is the region in the first quadrant bounded by $a \le r \le b$ and $0 \le \theta \le \pi/4$.

> Note $\frac{y^2}{x^2} = \tan^2\theta$, integrate θ then r, and simplify to $\frac{b^2-a^2}{2}\left(1-\frac{\pi}{4}\right)$.

General Change of Variables

> **Change-of-Variables Theorem:** For a C^1, one-to-one mapping $(u,v) \mapsto (x(u,v), y(u,v))$ from region S to D, let
> $J = \det\left[\partial(x,y)/\partial(u,v)\right]$. Then
> $$\iint_D f(x,y)\,dx\,dy = \iint_S f(x(u,v), y(u,v))\,|J(u,v)|\,du\,dv.$$

Example: Rectangle-to-Ellipse Mapping

Under $x = au$, $y = bv$, the unit disk $u^2 + v^2 \le 1$ maps to the ellipse $x^2/a^2 + y^2/b^2 \le 1$.

Final Summary & Takeaways

- Improper integrals handle unbounded domains or integrands; use iterated limits and comparison tests.
- Absolute convergence implies convergence of the original integral.
- A continuous function attains its average value in the domain.
- Polar coordinates simplify integration over circular regions, with area element $r\,dr\,d\theta$.
- The change-of-variables formula requires the Jacobian determinant.
- **Common Mistake:** Omitting the r factor in polar integrals or $|J|$ in general transformations.

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19. Triple Integrals & 3D Coordinate Transforms

Triple Integrals

Definition

> **Triple Integral:** For a bounded function $f(x, y, z)$ on a region $D \subset \mathbb{R}^3$,
>
> $$\iiint_D f(x, y, z)\, dV$$
>
> is the limit of Riemann sums partitioning D.

Properties

> **Linearity & Additivity:** Constants factor out; integrals over unions of nonoverlapping regions sum.
>
> **Symmetry:** If f is odd in one coordinate over a region symmetric about that coordinate-plane, the integral vanishes.

Fubini's Theorem

> **Iterated Integrals:** If f is continuous on a "simple" region D, then any order of integration applies:
>
> $$\iiint_D f\,dV = \int\int\int f(x,y,z)\,dz\,dy\,dx$$

Average Value & Center of Mass

> **Average Value:** $\overline{f} = \dfrac{1}{V(D)} \iiint_D f\,dV.$
>
> **Center of Mass (uniform density ρ):**
>
> $$\bar{x} = \frac{1}{V(D)} \iiint_D x\,dV, \quad \bar{y} = \frac{1}{V(D)} \iiint_D y\,dV, \quad \bar{z} = \frac{1}{V(D)} \iiint_D z\,dV.$$

Coordinate Transformations in 3D

Jacobian Determinant

> **Jacobian:** For $(u,v,w) \mapsto (x(u,v,w), y(u,v,w), z(u,v,w))$,
> $J = \det\left[\partial(x,y,z)/\partial(u,v,w)\right]$.
>
> **Change-of-Variables:**
>
> $$\iiint_D f(x,y,z)\,dV = \iiint_S f\left(x(u,v,w), y(u,v,w), z(u,v,w)\right)\left|J\right|\,du\,dv\,dw.$$

Cylindrical Coordinates

> **Transformation:**
>
> $$x = r\cos\theta,\ y = r\sin\theta,\ z = z.$$
>
> **Volume Element:** $dV = r\,dr\,d\theta\,dz.$

Spherical Coordinates

> **Transformation:**

> $$x = \rho \sin\phi \cos\theta,\ y = \rho \sin\phi \sin\theta,\ z = \rho \cos\phi.$$
>
> **Volume Element:** $dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta.$

Examples

Mass of a Tetrahedron

Region: Tetrahedron with vertices $(0,0,0), (1,0,0), (0,1,0), (0,0,1)$; density $\rho = 1$.

> **Express limits:** $0 \le x \le 1,\ 0 \le y \le 1 - x,\ 0 \le z \le 1 - x - y.$
>
> $$\iiint_D 1 \, dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz \, dy \, dx = \tfrac{1}{6}.$$

Volume of an Ellipsoid

Region: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} \le 1.$

> Use mapping $x = au,\ y = bv,\ z = cw$ on $u^2 + v^2 + w^2 \le 1$. Jacobian $= abc$.
>
> $$V = abc \iiint_{u^2+v^2+w^2 \le 1} dU = \tfrac{4\pi}{3} abc.$$

Final Summary & Takeaways

- Triple integrals compute volumes, masses, averages, and centers of mass via iterated integration.
- Jacobian determinants adjust the volume element under coordinate changes.
- Cylindrical and spherical coordinates simplify regions with rotational symmetry.
- **Common Mistake:** Omitting factors r in cylindrical or $\rho^2 \sin\phi$ in spherical integrals.

---

20. Vector Fields & Conservative Fields

Vector & Scalar Fields

> **SCALAR FIELD:** A function f assigning a real value to each point (x, y, z).
>
> **VECTOR FIELD:** A function $\mathbf{F}(x, y, z) = F_1\,\mathbf{i} + F_2\,\mathbf{j} + F_3\,\mathbf{k}$, where each F_i is a scalar field.

Common examples include

- Gravitational field of a point mass: $\mathbf{F} = -k\,\dfrac{\mathbf{r} - \mathbf{r}_0}{\|\mathbf{r} - \mathbf{r}_0\|^3}$.
- Velocity field of a steady rotating fluid: $\mathbf{v} = -\Omega y\,\mathbf{i} + \Omega x\,\mathbf{j}$.

Field Lines & Polar Representation

> **FIELD LINES:** Curves whose tangent at each point is parallel to the vector field there, satisfying
> $\dfrac{dx}{F_1} = \dfrac{dy}{F_2} = \dfrac{dz}{F_3}$.

For a plane field in polar form

$$\mathbf{F}(r,\theta) = F_r(r,\theta)\,\hat{r} \;+\; F_\theta(r,\theta)\,\hat{\theta}$$

with

$\hat{r} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j},\ \hat{\theta} = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}.$

Example: The rotating-field lines of $\mathbf{v} = \Omega(-y\,\mathbf{i} + x\,\mathbf{j})$ are circles $x^2 + y^2 = C$.

Conservative Fields & Exact Differentials

> **CONSERVATIVE FIELD:** $\mathbf{F} = \nabla\phi$ for some scalar potential ϕ.
>
> **EXACT DIFFERENTIAL:** An expression
>
> $$F_1\,dx + F_2\,dy + F_3\,dz$$
>
> is exact if it equals $d\phi$ for some ϕ.

Necessary Conditions

- In the plane:
  $\dfrac{\partial F_1}{\partial y} = \dfrac{\partial F_2}{\partial x}.$
- In space:

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \quad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \quad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}.$$

Finding a Potential

Integrate componentwise and match “constants” of integration:

Example: The gravitational point-mass field

$$\mathbf{F} = -k\frac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{(x^2 + y^2 + z^2)^{3/2}}$$

has potential $\phi = -\dfrac{k}{\sqrt{x^2 + y^2 + z^2}} + C.$

Equipotential Surfaces

> **EQUIPOTENTIAL SURFACES:** Level sets $\phi(x, y, z) = C$ of a potential function.
>
> Field lines intersect these surfaces at right angles, illustrating orthogonality of $\nabla\phi$ to level sets.

Final Takeaways

- Vector fields assign vectors to points; scalar fields assign scalars.
- Field lines follow $\frac{dx}{F_1} = \frac{dy}{F_2} = \frac{dz}{F_3}$.
- A field is conservative iff it equals a gradient; check mixed partials for exactness.
- Potentials are found by integrating components and ensuring consistency.
- Equipotential surfaces visualize scalar potentials and their orthogonality to field lines.

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21. Line Integrals of Scalar & Vector Fields

Line Integrals of Scalar Functions

> **Line Integral (scalar):** For a continuous $f(x, y, z)$ on a smooth curve $C\colon \mathbf{r}(t)$, $a \le t \le b$,
>
> $$\int_C f\,ds \;=\; \int_a^b f\big(\mathbf{r}(t)\big)\,\|\mathbf{r}'(t)\|\,dt.$$

Properties

- **Parametrization independent**: any smooth re-parametrization of C gives the same value.
- **Improper integrals**: allow unbounded curves or singular f.

Example (Arc-length weighted integral)

Compute $\displaystyle\int_C (x^2 + y^2)\,ds$ where C is the line from $(0, 0)$ to $(2, 1)$.

Parametrize $\mathbf{r}(t) = (2t, t)$, $0 \le t \le 1$. Then $\|\mathbf{r}'(t)\| = \sqrt{4+1} = \sqrt{5}$.

$$\int_C (x^2 + y^2)\,ds = \int_0^1 (4t^2 + t^2)\sqrt{5}\,dt = 5\sqrt{5}\int_0^1 t^2\,dt = \frac{5\sqrt{5}}{3}.$$

Line Integrals of Vector Fields

> **Work/Circulation:** For a vector field $\mathbf{F} = (F_1, F_2, F_3)$ along an oriented curve C,
>
> $$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \hat{\mathbf{T}}\, ds = \int_C F_1\, dx + F_2\, dy + F_3\, dz.$$

Evaluation via Parametrization

If $\mathbf{r}(t)$, $a \leq t \leq b$, parametrizes C,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}\big(\mathbf{r}(t)\big) \cdot \mathbf{r}'(t)\, dt.$$

Closed-Curve & Circulation

> **Circulation:** $\oint_C \mathbf{F} \cdot d\mathbf{r}$ measures net "work" around a closed path.

Examples

Example 1 (Non-conservative field):

$\mathbf{F} = (y,\ -x)$. Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the quarter-circle $x^2 + y^2 = 1$, $0 \leq \theta \leq \pi/2$.

Parametrize $\mathbf{r}(\theta) = (\cos\theta, \sin\theta)$. Then

$\mathbf{r}'(\theta) = (-\sin\theta, \cos\theta)$,

$\mathbf{F}\big(\mathbf{r}(\theta)\big) = (\sin\theta, -\cos\theta)$.

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi/2} (\sin\theta, -\cos\theta) \cdot (-\sin\theta, \cos\theta)\, d\theta = \int_0^{\pi/2} \big(-\sin^2\theta - \cos^2\theta\big)\, d\theta = -\frac{\pi}{2}.$$

Example 2 (Conservative field):

$\mathbf{F} = (2xy,\ x^2)$. Verify path-independence from $(0,0)$ to $(1,1)$.

$\partial F_1/\partial y = 2x = \partial F_2/\partial x \Rightarrow$ conservative.

Potential $\phi(x, y) = x^2 y$.

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \phi(1,1) - \phi(0,0) = 1.$$

Fundamental Theorem for Conservative Fields

> Theorem: On an open, connected domain D, the following are equivalent for smooth $\mathbf{F}$:
>
> 1. $\mathbf{F}$ is conservative ($\mathbf{F} = \nabla\phi$).

> 2. $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed $C \subset D$.
> 3. $\int_{P_0}^{P_1} \mathbf{F} \cdot d\mathbf{r}$ is path-independent.

Final Summary & Takeaways

- **Scalar integrals** $\int_C f\, ds$ weight by arc length.
- **Vector integrals** $\int_C \mathbf{F} \cdot d\mathbf{r}$ compute work or circulation.
- **Parametrization** reduces both to single-variable integrals.
- **Conservative fields** admit potentials; their line integrals depend only on endpoints.
- **Common Mistake:** Forgetting the Jacobian $\|\mathbf{r}'(t)\|$ in scalar integrals or sign/orientation in vector integrals.

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22. Surface Integrals & Flux

Parametric Surfaces

A **parametric surface** in $\mathbb{R}^3$ is given by

$$\mathbf{r}(u,v) = \big(x(u,v),\, y(u,v),\, z(u,v)\big), \quad (u,v) \in R,$$

where R is a region in the uv-plane. Each point of the surface corresponds uniquely to a (u, v).

Boundary of Parametric Surfaces

If $\mathbf{r}$ is one-to-one on R, the image of the boundary ∂R is the **boundary curve** of the surface. Traversing ∂R induces an orientation on this curve.

Composite Surfaces

When two (or more) parametric surfaces join along a common boundary curve—with matching parameterizations so normals agree—the union is a **composite surface**.

Surface Integrals

For a scalar function $f(x, y, z)$ on a smooth surface S, the **surface integral** is

$$\iint_S f\,dS = \iint_R f\big(\mathbf{r}(u,v)\big)\,\big\|\mathbf{r}_u \times \mathbf{r}_v\big\|\,du\,dv,$$

where $\mathbf{r}_u$ and $\mathbf{r}_v$ are partial derivatives.

Smooth Surfaces, Normals, and Area Elements

A surface is **smooth** if it has a unique tangent plane at each point. A normal vector is

$$\mathbf{n} = \mathbf{r}_u \times \mathbf{r}_v,$$

and the **area element** is

$$dS = \|\mathbf{n}\|\,du\,dv.$$

Oriented Surfaces

An **orientable surface** S admits a continuous unit normal field $\hat{\mathbf{N}}(P)$. Choosing $\hat{\mathbf{N}}$ defines a “positive side.”

- If S has boundary curve C, walking around C so that S stays on your left corresponds to the orientation induced by $\hat{\mathbf{N}}$.

Flux of a Vector Field Across an Oriented Surface

For a continuous vector field $\mathbf{F}$ and oriented surface S with unit normal $\hat{\mathbf{N}}$, the **flux** is

$$\iint_S \mathbf{F}\cdot\hat{\mathbf{N}}\,dS.$$

If S is given by $G(x,y,z)=0$ with $\nabla G \neq 0$ and projection onto the xy-plane, then

$$dS = \frac{\|\nabla G\|}{|G_z|}\,dx\,dy, \qquad \hat{\mathbf{N}} = \pm\frac{\nabla G}{\|\nabla G\|},$$

and signs chosen to match the desired orientation.

Examples

Example 1 (Surface Integral):

Compute $\iint_S z\,dS$ where S is the cone $z = \sqrt{x^2 + y^2}$, $0 \le z \le 1$.

- Parametrize by $x = u\cos v,\ y = u\sin v,\ z = u,\ 0 \le u \le 1,\ 0 \le v < 2\pi$.
- $\mathbf{r}_u \times \mathbf{r}_v$\ norm $= \sqrt{2}\,u$.
- Integral:

$$\int_0^{2\pi}\int_0^1 u\,(\sqrt{2}\,u)\,du\,dv = 2\pi\sqrt{2}\int_0^1 u^2\,du = \tfrac{2\pi\sqrt{2}}{3}.$$

Example 2 (Flux Integral):

Find the outward flux of $\mathbf{F} = (x, y, z)$ through the closed cylinder $x^2 + y^2 = a^2$, $-h \le z \le h$.

- Side: $\hat{\mathbf{N}} = (\cos\theta, \sin\theta, 0)$, $dS = a\,d\theta\,dz$, $\mathbf{F}\cdot\hat{\mathbf{N}} = a$.
- Flux through side:

$$\int_{-h}^{h}\int_0^{2\pi} a\,(a\,d\theta\,dz) = 2\pi a^2(2h).$$

- Top & bottom disks contribute each $\pi a^2 h$, so total flux

$$4\pi a^2 h + 2(\pi a^2 h) = 6\pi a^2 h.$$

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23. Gradient, Divergence, and Curl

The Gradient of a Scalar Field

- If $f(x, y, z)$ is a scalar function, its **gradient** is

$$\nabla f(x, y, z) = \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j} + \frac{\partial f}{\partial z}\,\mathbf{k}.$$

- We abbreviate the vector differential operator as

$$\nabla = \mathbf{i}\,\frac{\partial}{\partial x} + \mathbf{j}\,\frac{\partial}{\partial y} + \mathbf{k}\,\frac{\partial}{\partial z}.$$

Divergence and Curl of a Vector Field

- Let $\mathbf{F}(x, y, z) = F_1(x, y, z)\,\mathbf{i} + F_2(x, y, z)\,\mathbf{j} + F_3(x, y, z)\,\mathbf{k}$ be a vector field.
- The **divergence** of $\mathbf{F}$ is

$$\nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

- The **curl** of $\mathbf{F}$ is

$$\nabla \times \mathbf{F} \;=\; \Big(\frac{\partial F_3}{\partial y} \;-\; \frac{\partial F_2}{\partial z}\Big)\,\mathbf{i} \;+\; \Big(\frac{\partial F_1}{\partial z} \;-\; \frac{\partial F_3}{\partial x}\Big)\,\mathbf{j} \;+\; \Big(\frac{\partial F_2}{\partial x} \;-\; \frac{\partial F_1}{\partial y}\Big)\,\mathbf{k}.$$

  Equivalently, one can remember the determinant form:

$$\nabla \times \mathbf{F} \;=\; \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}.$$

Warning

> Do not confuse $\nabla \cdot \mathbf{F}$ (the divergence) with $\mathbf{F} \cdot \nabla$ (which is an operator acting on another function).

Interpretation of the Divergence

- At a point P , $\nabla \cdot \mathbf{F}$ measures the “net outward flux per unit volume” of $\mathbf{F}$ at P .
- Intuitively, if $\nabla \cdot \mathbf{F}(P) > 0$, the field is “spreading out” from P. If $\nabla \cdot \mathbf{F}(P) < 0$, the field is “converging” at P.

Interpretation of the Curl

- At a point P, $\nabla \times \mathbf{F}(P)$ measures the “local rotation” or “tendency to swirl” of $\mathbf{F}$ around P.
- If $\nabla \times \mathbf{F}(P) \neq \mathbf{0}$, the field has a nonzero infinitesimal circulation about P.

Irrotational Vector Fields

- A vector field $\mathbf{F} = F_1(x,y,z)\,\mathbf{i} + F_2(x,y,z)\,\mathbf{j} + F_3(x,y,z)\,\mathbf{k}$ is called **irrotational** if

$$\nabla \times \mathbf{F} \;=\; \mathbf{0}.$$

- Equivalently, $\mathbf{F}$ is irrotational if and only if

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \quad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \quad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}.$$

Simply Connected Domains

- A domain $D \subset \mathbb{R}^3$ is **simply connected** if it is connected and every simple closed curve in D can be continuously shrunk (homotoped) to a point without leaving D.
- In a simply connected domain, closed-loop integrals of an irrotational field must vanish.

Scalar Potentials (Conservative Fields)

> **Theorem (Existence of a Scalar Potential).**
>
> If $\mathbf{F}$ is a smooth, irrotational vector field on a simply connected domain $D \subset \mathbb{R}^3$, then there exists a scalar function $\phi(x, y, z)$ on D such that
>
> $$\mathbf{F} = \nabla\phi.$$
>
> In this case, ϕ is called a **potential function** for $\mathbf{F}$, and $\mathbf{F}$ is often said to be a **conservative** field.

Solenoidal Vector Fields

- A vector field $\mathbf{F}$ is called **solenoidal** if

  $$\nabla \cdot \mathbf{F} = 0$$

  everywhere in the domain.
- Equivalently, $\mathbf{F}$ has **zero divergence**.

Vector Potentials

> **Theorem (Existence of a Vector Potential).**
>
> If $\mathbf{F}$ is a smooth, solenoidal vector field on a domain D with the property that every closed surface in D bounds a region contained in D (for instance, any simply connected region in $\mathbb{R}^3$), then there exists a vector field $\mathbf{G}(x, y, z)$ on D such that

> $$\mathbf{F} \;=\; \nabla \times \mathbf{G}.$$
>
> In this situation, $\mathbf{G}$ is called a **vector potential** for $\mathbf{F}$.

- **Nonuniqueness of $\mathbf{G}$:**
  Since $\nabla \times (\nabla\phi) = \mathbf{0}$ for any smooth scalar ϕ, one may add any gradient field $\nabla\phi$ to $\mathbf{G}$ without changing $\nabla \times \mathbf{G}$. Hence, vector potentials are never unique.

Identities Involving Grad, Div, and Curl

Below are key vector-calculus identities (all assume sufficiently smooth functions and appropriate domains):

1. $\nabla \times (\nabla\phi) \;=\; \mathbf{0}$

   (The curl of any gradient field vanishes.)

2. $\nabla \cdot (\nabla \times \mathbf{G}) \;=\; 0$

   (The divergence of any curl field vanishes.)

3. $\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \Delta\mathbf{F},$

   where $\Delta\mathbf{F} = (\Delta F_1)\,\mathbf{i} + (\Delta F_2)\,\mathbf{j} + (\Delta F_3)\,\mathbf{k}$ is the vector Laplacian.

4. $\nabla \cdot (f\,\mathbf{F}) = f\,(\nabla \cdot \mathbf{F}) + \nabla f\;\cdot\;\mathbf{F}.$

5. $\nabla \times (f\,\mathbf{F}) = f\,(\nabla \times \mathbf{F}) + \nabla f \times \mathbf{F}.$

6. $\nabla(\mathbf{F} \cdot \mathbf{G}) = (\mathbf{F} \cdot \nabla)\,\mathbf{G} \;+\; (\mathbf{G} \cdot \nabla)\,\mathbf{F} \;+\; \mathbf{F} \times (\nabla \times \mathbf{G}) \;+\; \mathbf{G} \times (\nabla \times \mathbf{F}).$

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24. Green's, Stokes's and Divergence Theorem Theorem

Green's Theorem in the Plane

Introduction

Green's Theorem connects a line integral around a simple closed curve C in the plane with a double integral over the region R bounded by C. It is a special case of Stokes's Theorem in two dimensions.

Preliminaries

- **Region R:** A region $R \subset \mathbb{R}^2$ is called **simple** if it can be described as
  - x **-simple**: $\{(x, y) : a \leq x \leq b,\ g_1(x) \leq y \leq g_2(x)\}$, or
  - y**-simple**: $\{(x, y) : c \leq y \leq d,\ h_1(y) \leq x \leq h_2(y)\}.$
- **Piecewise-smooth boundary C:** Denote by C the positively oriented (counterclockwise) boundary of R.

Statement of Green's Theorem

Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane, and let R be the region enclosed by C . If

$$\mathbf{F}(x,y) = P(x,y)\,\mathbf{i} \;+\; Q(x,y)\,\mathbf{j}$$

has continuous partial derivatives $P_x,\ P_y,\ Q_x,\ Q_y$ on an open region containing R, then

$$\oint_C \bigl(P\,dx + Q\,dy\bigr) \;=\; \iint_R \Bigl(\frac{\partial Q}{\partial x} \;-\; \frac{\partial P}{\partial y}\Bigr)\,dA.$$

Interpretation

- The left side is the circulation of $\mathbf{F}$ around C.
- The right side is the signed area integral of the “curl” component $\bigl(Q_x - P_y\bigr)$ over R.
- Intuitively, Green’s Theorem says that the net “rotation” of $\mathbf{F}$ inside R equals the total line integral around the boundary.

Sketch of Proof

1. **Divide R into simple subregions:**
   - Decompose R into finitely many x-simple or y-simple regions.
   - Prove the theorem on each subregion, using the Fundamental Theorem of Calculus to convert the line integral to a double integral.
2. **Add up contributions:**
   - Boundary integrals on interior edges cancel in pairs (opposite orientations).
   - Only the outer boundary C remains, yielding the stated equality.

Examples

- **Rectangle Example:**

  Let $R = [a,b]\times[c,d]$ and $\mathbf{F} = (P,Q)$. Then

$$\oint_C (P\,dx + Q\,dy) = \int_{x=a}^{b} \bigl[P(x,d) - P(x,c)\bigr]\,dx \;+\; \int_{y=c}^{d} \bigl[Q(b,y) - Q(a,y)\bigr]\,dy,$$

  which matches

$$\iint_R (Q_x - P_y)\,dA = \int_{x=a}^{b} \int_{y=c}^{d} \big(Q_x(x,y) - P_y(x,y)\big)\,dy\,dx.$$

- **Area via Green's Theorem:**

  To compute $\mathrm{Area}(R)$, choose $\mathbf{F} = \left(-\frac{y}{2},\ \frac{x}{2}\right)$. Then $Q_x - P_y = 1$, so

  $$\mathrm{Area}(R) = \iint_R 1\,dA = \oint_C \left(-\tfrac{y}{2}\,dx + \tfrac{x}{2}\,dy\right).$$

The Divergence Theorem

The Two-Dimensional Divergence Theorem (Green's Flux Form)

Let D be a region in the plane with positively oriented, piecewise-smooth boundary C. If

$$\mathbf{F}(x,y) = P(x,y)\,\mathbf{i} + Q(x,y)\,\mathbf{j}$$

has continuous partials on an open set containing D, then

$$\oint_C \big(P\,dy - Q\,dx\big) \;=\; \iint_D \big(P_x + Q_y\big)\,dA.$$

Equivalently, in flux form,

$$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \iint_D \nabla\cdot\mathbf{F}\,dA,$$

where $\mathbf{n}$ is the outward-pointing unit normal to C.

Regular Domains in $\mathbb{R}^3$

A three-dimensional domain $D \subset \mathbb{R}^3$ is called **regular** if it can be written as a finite union of nonoverlapping subregions, each of which is simultaneously:

- x **-simple:** Each line parallel to the x-axis intersects the subregion in at most two points.
- y **-simple:** Each line parallel to the y-axis intersects the subregion in at most two points.
- z**-simple:** Each line parallel to the z-axis intersects the subregion in at most two points.

The Three-Dimensional Divergence Theorem

Let D be a bounded, regular domain in $\mathbb{R}^3$ with piecewise-smooth boundary surface S, oriented by the outward unit normal $\mathbf{n}$. If

$$\mathbf{F}(x,y,z) = F_1(x,y,z)\,\mathbf{i} \;+\; F_2(x,y,z)\,\mathbf{j} \;+\; F_3(x,y,z)\,\mathbf{k}$$

has continuously differentiable components on an open set containing D, then

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS \;=\; \iiint_D \nabla\cdot\mathbf{F}\,dV,$$

where $\nabla\cdot\mathbf{F} = F_{1_x} + F_{2_y} + F_{3_z}$.

Interpretation

- The left-hand side is the total outward **flux** of $\mathbf{F}$ through the closed surface S.
- The right-hand side is the triple integral of the divergence $\nabla\cdot\mathbf{F}$ over the volume D.
- Intuitively, the net “source strength” inside D equals the net flux out of D.

Variants and Consequences

- **Constant Vector Field:**
  If $\mathbf{c}$ is a constant vector, then $\nabla\cdot(\mathbf{F}\times\mathbf{c}) = 0$. Applying the Divergence Theorem to $\mathbf{F}\times\mathbf{c}$ yields identities involving surface integrals of cross products.
- **Scalar Times a Constant Vector:**
  If $\phi(x,y,z)$ is a scalar function and $\mathbf{c}$ is constant, then
  $\nabla\cdot(\phi\,\mathbf{c}) = \mathbf{c}\cdot\nabla\phi$. One can derive flux identities by applying the Divergence Theorem to $\phi\,\mathbf{c}$.

Examples

- **Flux Through a Cylinder:**

  Let D be the solid cylinder $x^2+y^2 \le a^2,\ 0\le z\le h$, and $\mathbf{F}(x,y,z) = (x,y,2z)$. Compute
  $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$.
  - Divergence: $\nabla\cdot\mathbf{F} = 1+1+2 = 4$.
  - Volume integral: $\iiint_D 4\,dV = 4\big(\text{Volume of } D\big) = 4\pi a^2 h$.
  - Hence, flux $= 4\pi a^2 h$.

- **Sphere:**

  For D the ball $x^2 + y^2 + z^2 \le R^2$ and $\mathbf{F} = (x, y, z)$,

  $\nabla \cdot \mathbf{F} = 3$.

  $\iiint_D 3\,dV = 3 \cdot \frac{4}{3}\pi R^3 = 4\pi R^3$.

  By the Divergence Theorem, flux through the sphere is $4\pi R^3$.

Stokes's Theorem

Introduction

Stokes's Theorem generalizes Green's Theorem to surfaces in $\mathbb{R}^3$. It relates the surface integral of the curl of a vector field over a surface S to the line integral of the field around the boundary curve ∂S.

Preliminaries

- **Surface S:** A piecewise-smooth, oriented surface in $\mathbb{R}^3$.
- **Boundary Curve ∂S:** The (possibly) closed, piecewise-smooth curve bounding S, oriented consistently with S (right-hand rule).
- **Unit Normal $\mathbf{n}$:** Choose a continuous unit normal vector field $\mathbf{n}(x, y, z)$ on S.
- **Vector Field:** Let

  $$\mathbf{F}(x, y, z) = F_1\,\mathbf{i} \;+\; F_2\,\mathbf{j} \;+\; F_3\,\mathbf{k},$$

  with continuous partial derivatives on an open set containing S.

Statement of Stokes's Theorem

If S is oriented by the unit normal $\mathbf{n}$ and ∂S is given the induced positive orientation (right-hand rule), then

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS \;=\; \oint_{\partial S} \mathbf{F} \cdot d\mathbf{r}.$$

- **Left-hand side:** Surface integral of the normal component of $\nabla \times \mathbf{F}$.
- **Right-hand side:** Circulation of $\mathbf{F}$ around the boundary ∂S.

Orientation Convention

Use the **right-hand rule**: Curl the fingers of your right hand in the direction of traversal around ∂S; your thumb points in the direction of the chosen normal $\mathbf{n}$.

Interpretation

- The integral of "local rotation" (curl) over the entire surface equals the total "circulation" along the boundary curve.
- If $\nabla \times \mathbf{F} = \mathbf{0}$ on S, then $\oint_{\partial S} \mathbf{F} \cdot d\mathbf{r} = 0$. This generalizes the fact that a conservative field has zero circulation around any closed loop.

Examples

- **Graph of a Function:**

  Let S be the graph $z = g(x, y)$ over a domain D in the xy-plane. Then one can express $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$ as a double integral over D, and ∂S projects to ∂D.

- **Flat Disk in Plane:**

  Take S to be the disk $x^2 + y^2 \leq a^2$ in the xy-plane (oriented upward, $\mathbf{n} = \mathbf{k}$). For $\mathbf{F} = (P, Q, R)$,

  $$(\nabla \times \mathbf{F}) \cdot \mathbf{k} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}.$$

  Stokes's Theorem reduces to Green's Theorem:

  $$\iint_S (Q_x - P_y)\, dA = \oint_{x^2+y^2=a^2} \big(P\, dx + Q\, dy\big).$$

- **Half-Sphere:**

  Let S be the upper hemisphere $x^2 + y^2 + z^2 = R^2,\ z \geq 0$. Its boundary ∂S is the circle $x^2 + y^2 = R^2,\ z = 0$. For $\mathbf{F} = (-y,\ x,\ 0)$, one checks $\nabla \times \mathbf{F} = (0, 0, 2)$. Then

  $$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \iint_S 2\, dS = 2 \cdot \big(\text{Area of hemisphere}\big) = 2 \cdot 2\pi R^2 = 4\pi R^2.$$

  On the other hand,

$$\oint_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \oint_{x^2+y^2=R^2} \big(-y\,dx + x\,dy\big) = 4\pi R^2,$$

consistent with Stokes's Theorem.

Generalized Stokes's Theorem (Outline)

On an oriented n-dimensional manifold M with boundary ∂M, if ω is a smooth $(n-1)$-form with compact support, then

$$\int_M d\omega \;=\; \int_{\partial M} \omega.$$

- Here, d is the exterior derivative, and integration is taken with respect to the induced orientations.
- Green's Theorem, the Divergence Theorem, and Stokes's Theorem are all special cases of this general result.

Summary of Key Relationships

- **Green's Theorem (2D)**

$$\oint_C (P\,dx + Q\,dy) = \iint_R \big(Q_x - P_y\big)\,dA.$$

- **Divergence Theorem (3D Flux Form)**

$$\iint_S \mathbf{F} \cdot \mathbf{n}\,dS = \iiint_D \nabla \cdot \mathbf{F}\,dV.$$

- **Stokes's Theorem (Surface–Curve)**

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\,dS = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{r}.$$