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PHYS101

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1. Units and Vectors

Units and Vectors

Understanding physical quantities is fundamental in science and engineering. A **unit** is a standard quantity used to specify measurements, while a **vector** is a quantity that has both magnitude and direction. Although these topics are distinct, they often come together in problems where both the size of a quantity (its unit) and its direction matter (as in displacement or force).



SI Units

The International System of Units (SI) is the modern form of the metric system. It defines standard units for various physical quantities:

- **Mass:** kilogram (kg)
- **Time:** second (s)
- **Length:** meter (m)
- **Force:** newton (N)
- **Electric Current:** ampere (A)
- **Temperature:** kelvin (K)

SI UNITS: The standardized set of measurements defined by international consensus; for example, the kilogram is defined using the Planck constant, and the meter is defined based on the distance traveled by light in a vacuum.

SI Base Units			
Base quantity		Base unit	
Name	Typical symbol	Name	Symbol
time	t	second	s
length	$l, x, r, \text{etc.}$	meter	m
mass	m	kilogram	kg
electric current	I, i	ampere	A
thermodynamic temperature	T	kelvin	K
amount of substance	n	mole	mol
luminous intensity	I_v	candela	cd

Source: NIST Special Publication 330:2019, Table 2.

Example Calculation Using SI Units

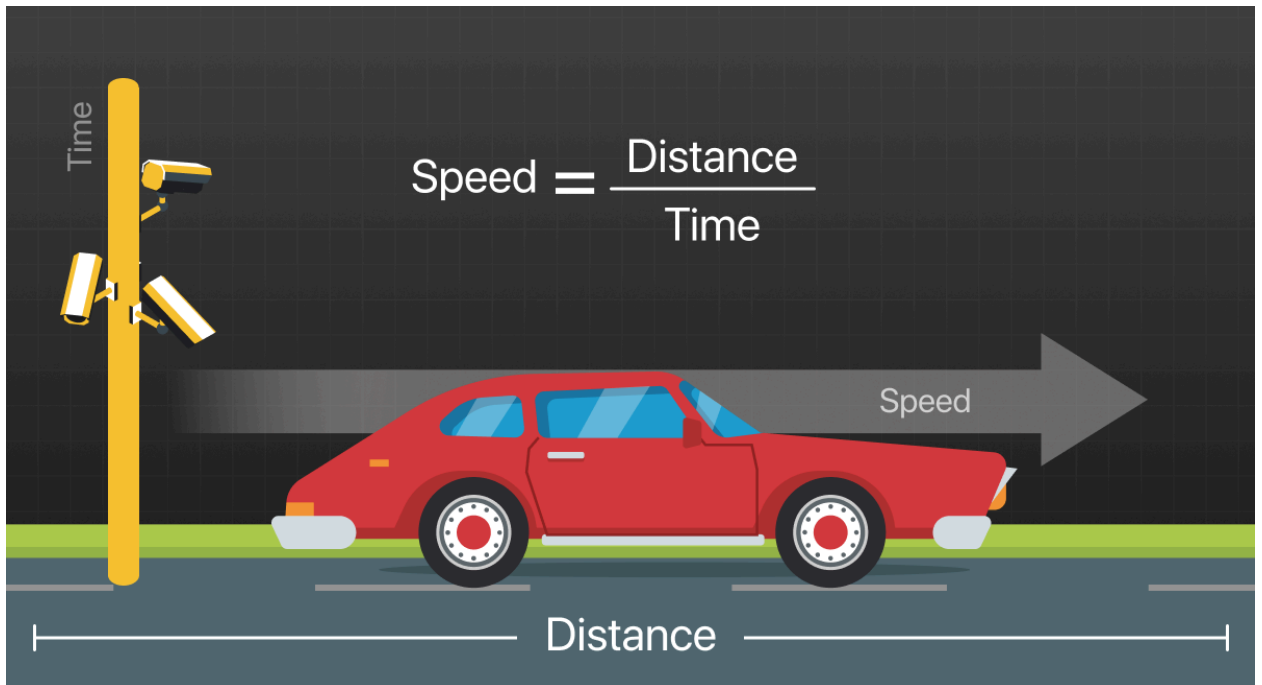
Consider a car moving at 80 km/h. To calculate the distance it travels in 10 seconds:

1. **Convert speed to m/s:**

$$80 \text{ km/h} = 80 \times \frac{1000 \text{ m}}{3600 \text{ s}} \approx 22.22 \text{ m/s}$$

2. **Calculate distance:**

$$\text{Distance} = \text{speed} \times \text{time} = 22.22 \text{ m/s} \times 10 \text{ s} \approx 222.2 \text{ m}$$

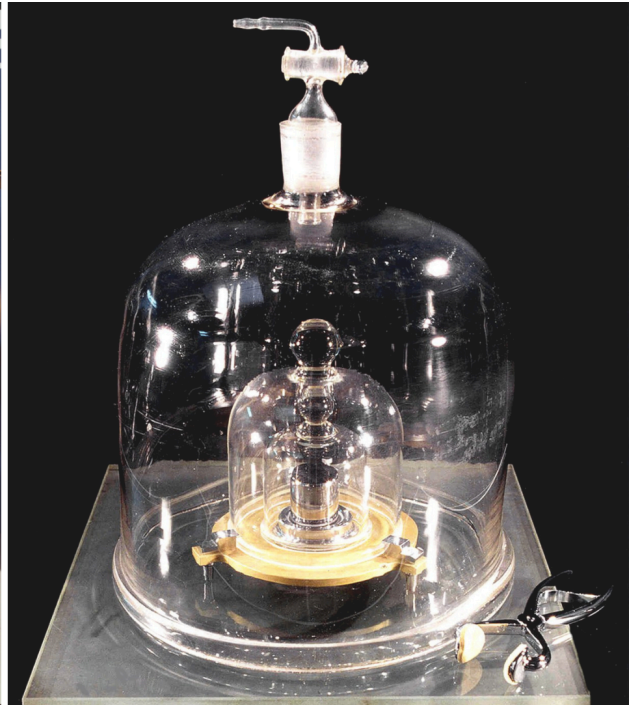
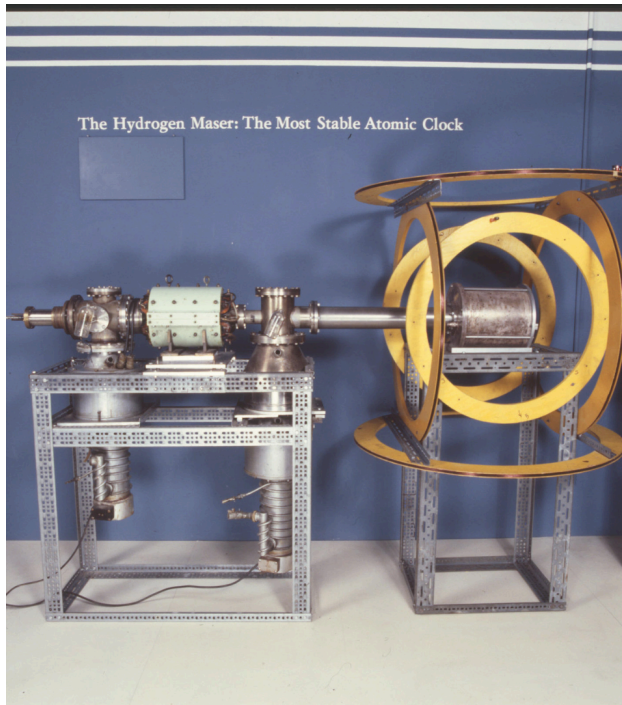


Defining SI Units

SI units are defined using physical constants and precise measurement techniques:

- **Mass:** Historically defined by the international prototype kilogram, now redefined using the Planck constant.
- **Length:** Defined as the distance light travels in a vacuum in $1/299,792,458$ seconds.
- **Time:** Defined by atomic clocks, particularly using the frequency of radiation from cesium-133 atoms.
- **Electric Current:** Defined via the elementary charge and other fundamental constants.

DEFINING SI UNITS: SI units are grounded in invariant physical constants, ensuring universal reproducibility and precision.



Scientific Notation and Order of Magnitude

Scientific notation is a compact way to express very large or very small numbers. For example:

- 3,000,000 is written as 3×10^6 .

Prefixes in SI Notation:

- **Submultiples:**
 10^{-3} : milli, 10^{-6} : micro, 10^{-9} : nano, 10^{-12} : pico, 10^{-15} : femto
- **Multiples:**
 10^3 : kilo, 10^6 : mega, 10^9 : giga, 10^{12} : tera

SCIENTIFIC NOTATION: A method to express numbers as a product of a coefficient and a power of ten, which simplifies the comparison of orders of magnitude.

Powers of 10

Prefix	Exponent	Number	Scientific Notation	Name
Exa (E)	18	1,000,000,000,000,000,000	10^{18}	quintillion
Peta (P)	15	1,000,000,000,000,000	10^{15}	quadrillion
Tera (T)	12	1,000,000,000,000	10^{12}	trillion
Giga (G)	9	1,000,000,000	10^9	billion
Mega (M)	6	1,000,000	10^6	million
kilo (k)	3	1,000	10^3	thousand
hecto (h)	2	100	10^2	hundred
deca (da)	1	10	10^1	ten
	0	1	10^0	one
deci (d)	-1	0.1	10^{-1}	one tenth
centi (c)	-2	0.01	10^{-2}	one hundredth
milli (m)	-3	0.001	10^{-3}	one thousandth
micro (μ)	-6	0.000001	10^{-6}	one millionth
nano (n)	-9	0.000000001	10^{-9}	one billionth
pico (p)	-12	0.000000000001	10^{-12}	one trillionth
femto (f)	-15	0.000000000000001	10^{-15}	one quadrillionth
atto (a)	-18	0.000000000000000001	10^{-18}	one quintillionth

Significant Figures and Uncertainty

Measurements carry uncertainty, and the number of significant figures indicates the precision of a measurement. For example, "5 meter" and "5.0 meter" are not equivalent in a measurement context:

- **5 meter** implies one significant figure.
- **5.0 meter** implies two significant figures.

Finding the Number of Significant Figures

- **Rule 1:** Non-zero digits are always significant.
- **Rule 2:** Zeros between non-zero digits are significant.
- **Rule 3:** Leading zeros are not significant.
- **Rule 4:** Trailing zeros are significant only if there is a decimal point.

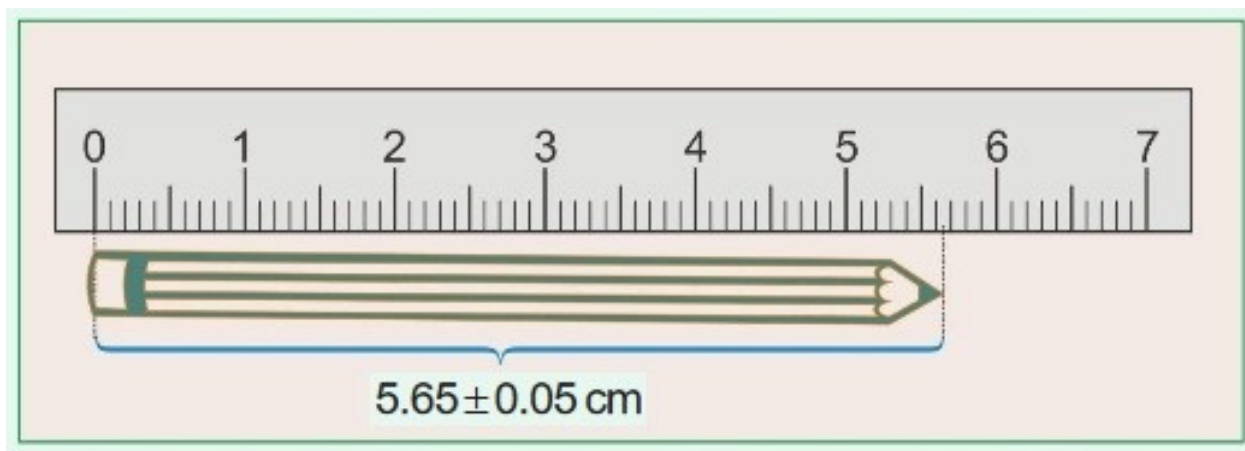
SIGNIFICANT FIGURES: The digits in a measurement that carry meaning contributing to its precision.

Importance of Significant Figures

The precision of a measurement affects how results are reported, especially when combining measurements:

- **Addition/Subtraction:** The result should be reported with the same number of **decimal places** as the measurement with the fewest decimal places.
 - **Example:** $5 + 4.92 = 9.92$
 - **5 is treated as an integer (no decimal places)**, then the result should be rounded to **10** (not 9).
- **Multiplication/Division:** The result should have as many **significant figures** as the measurement with the fewest significant figures.
 - **Example:** $1.2 \times 50 \times 3 = 180$
 - If **50 has only 1 significant figure**, then the result should be rounded to **200** (1 significant figure).

UNCERTAINTY: The degree of doubt about a measurement's exact value, expressed through significant figures.



Precision and Accuracy

Precision refers to the consistency or repeatability of measurements, while **accuracy** indicates how close a measurement is to the true or accepted value.

- **Precision:** High precision means repeated measurements yield similar results.

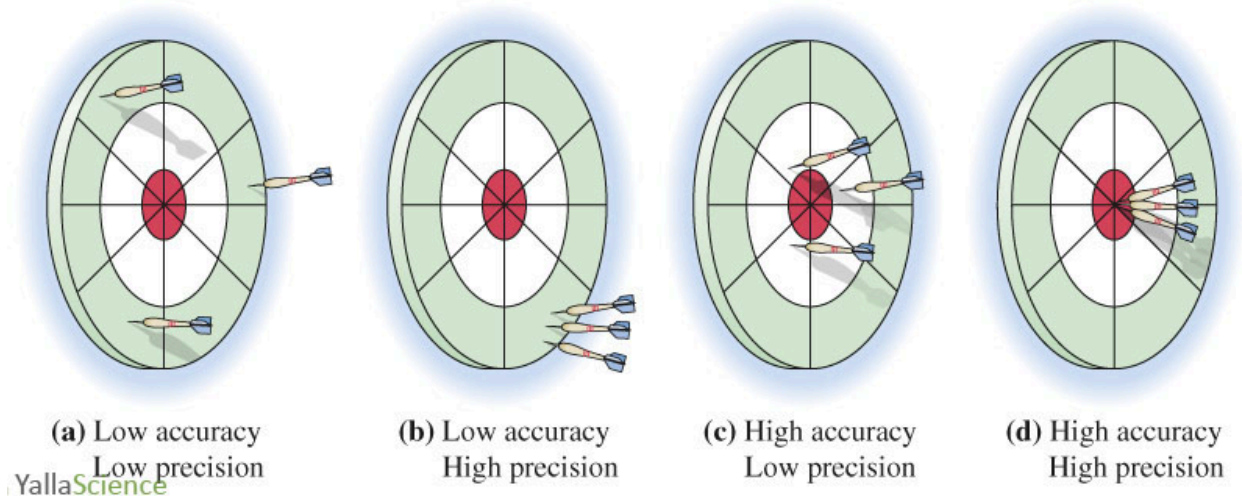
- **Accuracy:** High accuracy means the measurement is close to the true value.

Example:

- A set of darts hitting the same spot repeatedly (high precision) but far from the bullseye (low accuracy).
- Alternatively, darts that scatter around the bullseye (high accuracy, low precision).

PRECISION: The consistency of repeated measurements.

ACCURACY: How close a measurement is to the true value.



Summary

In this lecture, we covered:

- **Units and Vectors:** Basic concepts of measurements and quantities with direction.
- **SI Units:** Standardized units for mass, time, length, force, current, and temperature, along with their definitions.
- **Scientific Notation and Order of Magnitude:** Expressing numbers in a compact form and understanding SI prefixes.
- **Significant Figures and Uncertainty:** How measurement precision is indicated and how to correctly propagate uncertainty in results.
- **Precision and Accuracy:** Distinguishing between the consistency of measurements and their closeness to the true value.

Understanding these concepts is critical in physics and engineering, as they lay the foundation for accurate measurement, reporting, and further quantitative analysis.

Self Test

 [Self-Test: Lecture 1](#)

Raw Notes

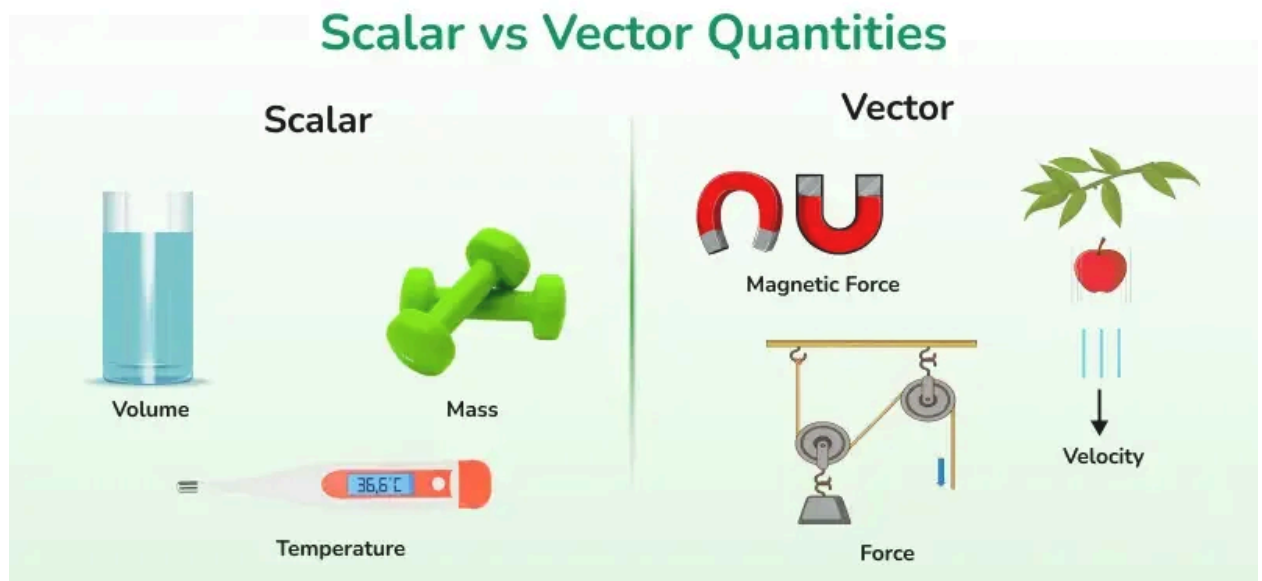
 [Raw Notes](#)



2. Units and Vectors - Extended

Introduction to Vectors and Scalars

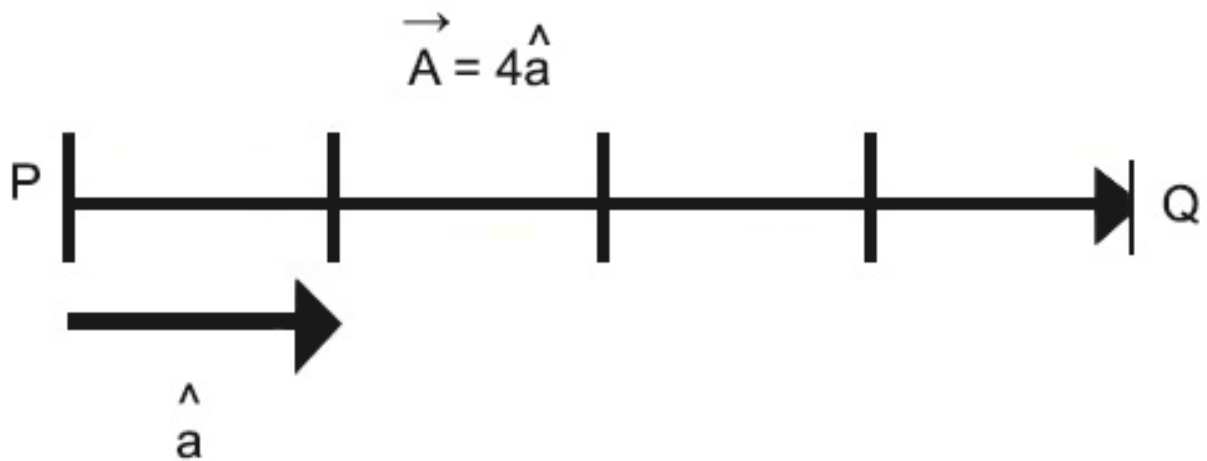
In many fields such as physics, engineering, and computer science, it is essential to distinguish between two fundamental types of quantities: **scalars** and **vectors**. A scalar is a quantity described solely by a magnitude (a numerical value), whereas a vector is defined by both a magnitude and a direction. For example, temperature (a scalar) only has a value (e.g., 25°C), while displacement (a vector) has both a length and a direction.



What is a Vector?

A vector is a mathematical entity that has both a magnitude and a direction. We denote the magnitude of a vector $\mathbf{A}$ as $|\mathbf{A}|$, and its direction is usually indicated by a unit vector. Two vectors are considered equal if they have the same magnitude and point in the same direction. When a vector is multiplied by a scalar (a real number), its magnitude is scaled by that number, while its direction remains unchanged.

VECTOR: A quantity with both magnitude and direction, represented as $\mathbf{A} = |\mathbf{A}|\hat{A}$, where $\hat{A}$ is the unit vector indicating direction.



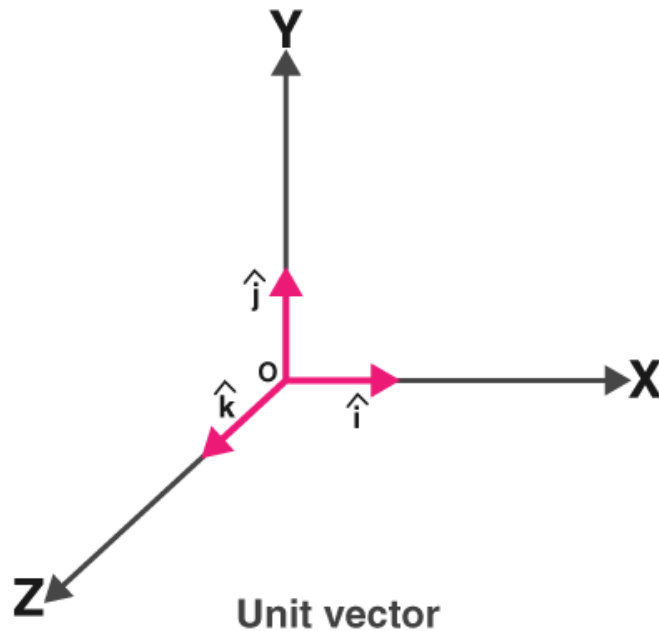
Unit Vectors and Vector Representation

Unit vectors are used to indicate direction. A unit vector has a magnitude of 1 and is typically denoted by symbols such as $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ for the x , y , and z axes respectively. Any vector in three-dimensional space can be written as the product of its magnitude and a unit vector in its direction:

$$\mathbf{A} = |\mathbf{A}|\hat{A}$$

or expressed in component form as:

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$$



Vector Operations: Addition and Subtraction

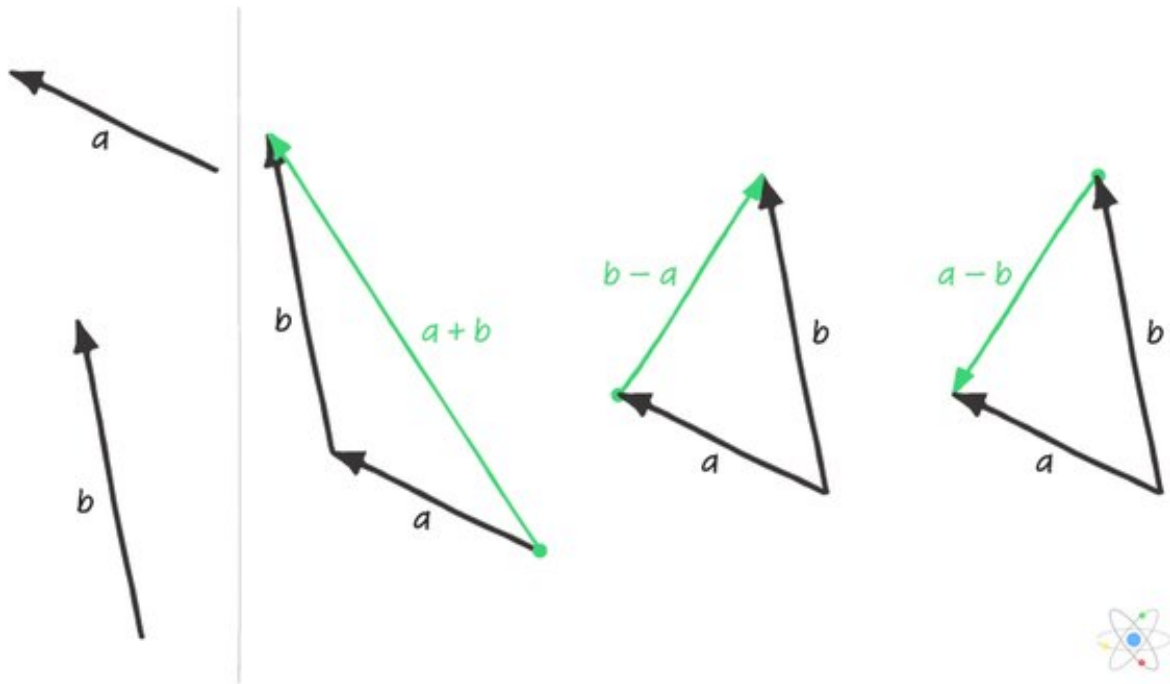
Vectors can be added together and subtracted from one another. The rules for vector addition are:

- **Commutative Property:** $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$
- **Associative Property:** $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$

Subtraction of vectors is essentially the addition of the negative. That is,

$$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B})$$

where $-\mathbf{B}$ is a vector with the same magnitude as $\mathbf{B}$ but opposite direction.



Components of a Vector and Vector Notation

Any vector in the plane can be decomposed into its components along the x and y axes.

For a vector $\mathbf{R}$:

$$\mathbf{R} = R_x \mathbf{i} + R_y \mathbf{j}$$

The magnitude of $\mathbf{R}$ is calculated by:

$$|\mathbf{R}| = \sqrt{R_x^2 + R_y^2}$$

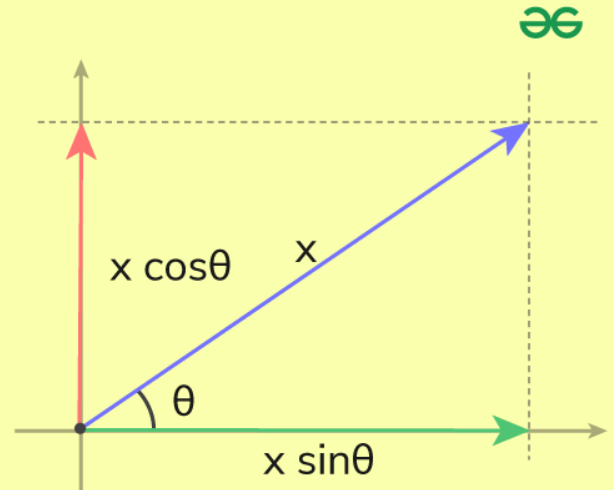
In three dimensions, the formula becomes:

$$|\mathbf{R}| = \sqrt{R_x^2 + R_y^2 + R_z^2}$$

The direction (angle θ) with respect to the x -axis is given by:

$$\theta = \arctan \left(\frac{R_y}{R_x} \right)$$

Components of a Vector



Example: Computing the Magnitude of a Vector Expression

Consider two vectors in three dimensions:

$$\mathbf{D} = 6\mathbf{i} + 3\mathbf{j} - \mathbf{k} \quad \text{and} \quad \mathbf{E} = 4\mathbf{i} - 5\mathbf{j} + 8\mathbf{k}$$

We wish to compute the magnitude of $2\mathbf{D} - \mathbf{E}$.

First, calculate:

$$2\mathbf{D} = 2(6\mathbf{i} + 3\mathbf{j} - \mathbf{k}) = 12\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}$$

Then,

$$2\mathbf{D} - \mathbf{E} = (12\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}) - (4\mathbf{i} - 5\mathbf{j} + 8\mathbf{k}) = (12 - 4)\mathbf{i} + (6 - (-5))\mathbf{j} + (-2 - 8)\mathbf{k} = 8\mathbf{i} + 11\mathbf{j} - 10\mathbf{k}$$

The magnitude is:

$$|\mathbf{R}| = \sqrt{8^2 + 11^2 + (-10)^2} = \sqrt{64 + 121 + 100} = \sqrt{285} \approx 16.88$$

Scalar and Vector Multiplication

Multiplying by a Scalar

Multiplying a vector by a scalar changes its magnitude without altering its direction:

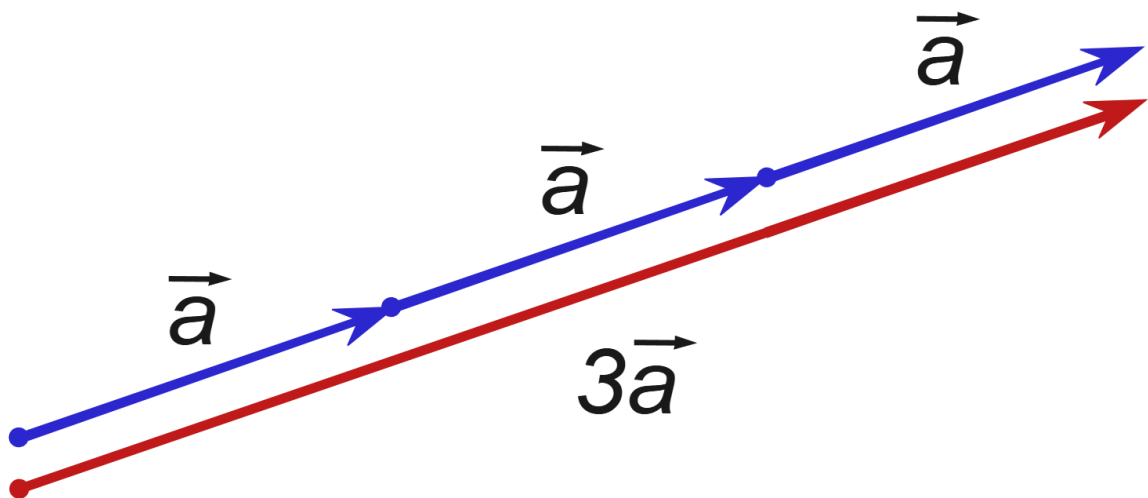
$$c\mathbf{A} = (c \cdot A_x)\mathbf{i} + (c \cdot A_y)\mathbf{j} + (c \cdot A_z)\mathbf{k}$$

If $c > 1$, the vector stretches; if $0 < c < 1$, it contracts; if $c < 0$, it reverses direction.

Multiplication Rules Summary

- **Scalar × Scalar:** Results in a scalar.
- **Scalar × Vector:** Results in a vector.
- **Vector × Vector:** Can yield either a scalar (dot product) or a vector (cross product), depending on the operator.

MULTIPLICATION RULES: The type of multiplication (scalar or vector product) determines whether the result is a scalar or a vector.



Dot Product (Scalar Product)

The dot product of two vectors $\mathbf{A}$ and $\mathbf{B}$ is defined as:

$$\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z = |\mathbf{A}| |\mathbf{B}| \cos \theta$$

where θ is the smaller angle between the two vectors.

Properties:

- **Commutative:** $\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$
- Provides a measure of how much one vector extends in the direction of another.
- Used to determine the angle between vectors:

$$\theta = \arccos \left(\frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}| |\mathbf{B}|} \right)$$

Example Calculation:

Given:

$$\mathbf{A} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}, \quad \mathbf{B} = -4\mathbf{i} + 2\mathbf{j} - \mathbf{k}$$

Compute the dot product:

$$\mathbf{A} \cdot \mathbf{B} = (2 \times -4) + (3 \times 2) + (1 \times -1) = -8 + 6 - 1 = -3$$

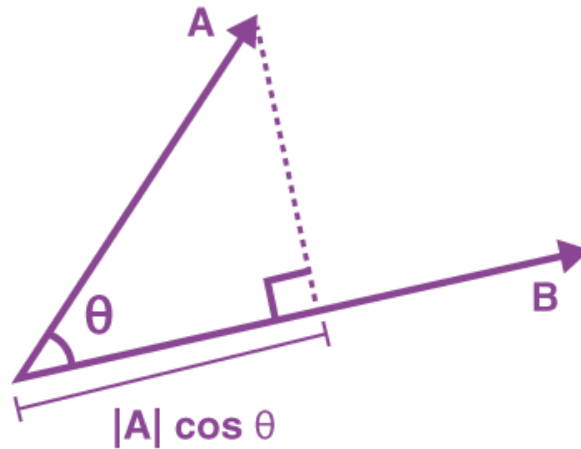
Find magnitudes:

$$|\mathbf{A}| = \sqrt{2^2 + 3^2 + 1^2} = \sqrt{4 + 9 + 1} = \sqrt{14}$$

$$|\mathbf{B}| = \sqrt{(-4)^2 + 2^2 + (-1)^2} = \sqrt{16 + 4 + 1} = \sqrt{21}$$

Thus,

$$\cos \theta = \frac{-3}{\sqrt{14}\sqrt{21}}, \quad \theta = \arccos \left(\frac{-3}{\sqrt{14}\sqrt{21}} \right)$$



Cross Product (Vector Product)

The cross product of two vectors **A** and **B** results in a vector **C** that is perpendicular to both **A** and **B**. It is defined as:

$$\mathbf{A} \times \mathbf{B} = (A_y B_z - A_z B_y)\mathbf{i} + (A_z B_x - A_x B_z)\mathbf{j} + (A_x B_y - A_y B_x)\mathbf{k}$$

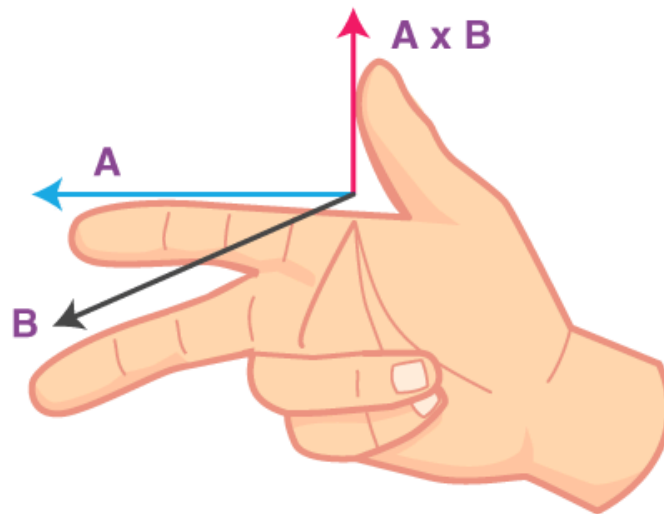
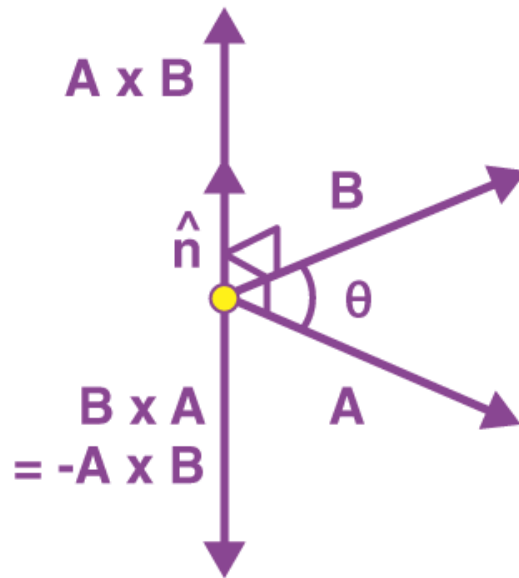
with magnitude:

$$|\mathbf{A} \times \mathbf{B}| = |\mathbf{A}| |\mathbf{B}| \sin \theta$$

where θ is the smaller angle between **A** and **B**.

Properties:

- **Anticommutative:** $\mathbf{A} \times \mathbf{B} = -(\mathbf{B} \times \mathbf{A})$
- The resulting vector is orthogonal (perpendicular) to both **A** and **B**.
- If the vectors are parallel, $\sin \theta = 0$ and hence $\mathbf{A} \times \mathbf{B} = \mathbf{0}$.



Summary of Vector Operations

- **Addition/Subtraction:** Vectors add component-wise; subtraction is addition of the negative.

$$\mathbf{A} + \mathbf{B} = (A_x + B_x)\mathbf{i} + (A_y + B_y)\mathbf{j} + (A_z + B_z)\mathbf{k}$$

Properties such as commutativity and associativity hold.

- **Scalar Multiplication:** Multiplying by a scalar changes the magnitude of the vector.

$$c\mathbf{A} = (cA_x)\mathbf{i} + (cA_y)\mathbf{j} + (cA_z)\mathbf{k}$$

- **Dot Product:** Yields a scalar, used to determine the angle between vectors.
- **Cross Product:** Yields a vector perpendicular to the plane containing the original vectors.

Putting It All Together: Example Problem

Problem:

Given:

$$\mathbf{D} = 6\mathbf{i} + 3\mathbf{j} - \mathbf{k} \quad \text{and} \quad \mathbf{E} = 4\mathbf{i} - 5\mathbf{j} + 8\mathbf{k}$$

Find $|2\mathbf{D} - \mathbf{E}|$.

Solution:

1. **Compute $2\mathbf{D}$:**

$$2\mathbf{D} = 2(6\mathbf{i} + 3\mathbf{j} - \mathbf{k}) = 12\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}$$

2. **Subtract $\mathbf{E}$:**

$$\begin{aligned} 2\mathbf{D} - \mathbf{E} &= (12\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}) - (4\mathbf{i} - 5\mathbf{j} + 8\mathbf{k}) \\ &= (12 - 4)\mathbf{i} + (6 - (-5))\mathbf{j} + (-2 - 8)\mathbf{k} \\ &= 8\mathbf{i} + 11\mathbf{j} - 10\mathbf{k} \end{aligned}$$

3. **Calculate the Magnitude:**

$$|2\mathbf{D} - \mathbf{E}| = \sqrt{8^2 + 11^2 + (-10)^2} = \sqrt{64 + 121 + 100} = \sqrt{285} \approx 16.88$$

Summary

In this lecture note, we have covered:

- The basic definitions of **scalars** and **vectors**, emphasizing that vectors have both magnitude and direction.

- The use of **SI units** and the importance of standardization in measurement.
- The role of **unit vectors** in representing direction and how vectors can be expressed in component form.
- Fundamental operations such as vector **addition**, **subtraction**, and **scalar multiplication**.
- The methods to compute the **magnitude** and **direction** of a vector.
- Detailed explanations and examples of the **dot product** (yielding a scalar) and **cross product** (yielding a vector perpendicular to the given vectors).
- A comprehensive worked example illustrating these concepts in action.

Understanding these vector operations and properties is crucial for applications in physics, computer graphics, engineering, and many areas of science.

Self Test

✿ [Self-Test: Lecture 2](#)

Raw Notes

✿ [Raw Notes](#)



3. Kinematics in 1D

Motion in One Dimension

Motion in 1D involves understanding how a particle's position changes over time. The main quantities are:

POSITION: The location of an object along a line, represented as a vector x (in one dimension, this is simply a number) with units of meters (m).

DISPLACEMENT: The change in position, defined as the difference between the final and initial positions, i.e.,

$$\Delta x = x_2 - x_1$$

It is also measured in meters (m).

VELOCITY: The rate of change of position with respect to time. It is a vector, whose magnitude is given in meters per second (m/s), and its direction is the same as that of the displacement.

Average Velocity

Average velocity over a time interval is calculated as the displacement divided by the change in time:

$$\text{Average Velocity} = \frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1}$$

- **Unit:** meters per second (m/s)

Note: The displacement used here is the net change in position, not the total distance traveled.

Instantaneous Velocity

Instantaneous velocity is the velocity of an object at a specific moment in time. It is defined as the derivative of the position function $x(t)$ with respect to time:

$$v(t) = \lim_{\Delta t \rightarrow 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$$

- **Interpretation:** On an x - t graph, the instantaneous velocity at a point is the slope of the tangent line at that point.

Example: Motion in 1D

Consider the position function:

$$x(t) = 20 + 5t^2$$

where t is in seconds and x is in meters.

a) Displacement between $t = 1$ second and $t = 2$ seconds

- **At $t = 1$:**

$$x(1) = 20 + 5(1)^2 = 20 + 5 = 25 \text{ m}$$

- **At $t = 2$:**

$$x(2) = 20 + 5(2)^2 = 20 + 20 = 40 \text{ m}$$

- **Displacement:**

$$\Delta x = x(2) - x(1) = 40 - 25 = 15 \text{ m}$$

b) Average Velocity between $t = 1$ and $t = 2$

$$\text{Average Velocity} = \frac{\Delta x}{\Delta t} = \frac{15 \text{ m}}{2 - 1 \text{ s}} = 15 \text{ m/s}$$

c) Instantaneous Velocity at $t = 1$ and $t = 2$

- **Find the derivative:**

$$v(t) = \frac{dx}{dt} = \frac{d}{dt}(20 + 5t^2) = 10t$$

- **At $t = 1$:**

$$v(1) = 10(1) = 10 \text{ m/s}$$

- **At $t = 2$:**

$$v(2) = 10(2) = 20 \text{ m/s}$$

Acceleration

Acceleration is the rate of change of velocity with respect to time.

ACCELERATION: The change in velocity per unit time, calculated as:

$$a = \frac{\Delta v}{\Delta t}$$

Its unit is meters per second squared (m/s^2).

Instantaneous Acceleration

Instantaneous acceleration is the derivative of the velocity function with respect to time, or the second derivative of the position function:

$$a(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

Example: Acceleration Calculation

Consider the velocity function:

$$v(t) = 60 + 0.5t^2$$

a) Change in Velocity from $t = 1$ to $t = 3$

- **At $t = 1$:**

$$v(1) = 60 + 0.5(1)^2 = 60 + 0.5 = 60.5 \text{ m/s}$$

- **At $t = 3$:**

$$v(3) = 60 + 0.5(3)^2 = 60 + 4.5 = 64.5 \text{ m/s}$$

- **Change in Velocity:**

$$\Delta v = v(3) - v(1) = 64.5 - 60.5 = 4 \text{ m/s}$$

b) Average Acceleration between $t = 1$ and $t = 3$

$$\text{Average Acceleration} = \frac{\Delta v}{\Delta t} = \frac{4 \text{ m/s}}{3 - 1 \text{ s}} = \frac{4}{2} = 2 \text{ m/s}^2$$

c) Instantaneous Acceleration at $t = 1$ and $t = 3$

- **Find the derivative of $v(t)$:**

$$a(t) = \frac{dv}{dt} = \frac{d}{dt}(60 + 0.5t^2) = t$$

- **At $t = 1$:**

$$a(1) = 1 \text{ m/s}^2$$

- At $t = 3$:

$$a(3) = 3 \text{ m/s}^2$$

Summary

- **Position and Displacement:** Position is the location of an object; displacement is the change in position ($\Delta x = x_2 - x_1$).
- **Average Velocity:** Calculated as displacement over time, $\frac{\Delta x}{\Delta t}$.
- **Instantaneous Velocity:** The derivative of the position function with respect to time, $v(t) = \frac{dx}{dt}$; graphically, it is the slope of the tangent to the x - t curve.
- **Acceleration:** The rate of change of velocity, with average acceleration $\frac{\Delta v}{\Delta t}$ and instantaneous acceleration as $a(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2}$.

This material lays the foundation for understanding kinematics in one dimension and illustrates how calculus is used to derive instantaneous rates from average quantities.

Self Test

✿ [Self-Test: Lecture 3](#)

Raw Notes

✿ [Raw Notes](#)



4. Kinematics in 1D - Extended

Motion Along Straight Line

When an object moves so that its position changes with respect to time along a single axis (often the x -axis), this is called **1D motion**.

Reference Frames and Displacement

A **reference frame** provides the coordinate system and origin used to describe motion.

DISPLACEMENT: If x_1 is the initial position and x_2 is the final position, then

$$\Delta x = x_2 - x_1$$

Displacement is a vector quantity in 1D, and it can be positive or negative depending on direction.

Average Velocity

AVERAGE VELOCITY: The overall change in position per unit time,

$$v_{\text{avg}} = \frac{\Delta x}{\Delta t}$$

It represents the constant velocity that would take an object from x_1 to x_2 in the same time interval.

Instantaneous Velocity

INSTANTANEOUS VELOCITY: The velocity at a specific instant, given by the time derivative of position,

$$v(t) = \frac{dx(t)}{dt}$$

This tells us how fast an object is moving and in which direction at a particular moment.

Speed vs Velocity

- **Speed:** A scalar, the magnitude of velocity (always non-negative).
- **Velocity:** A vector in 1D, can be positive or negative indicating direction.

SPEED: If $v(t)$ is the velocity, speed is $|v(t)|$.

Average and Instantaneous Acceleration

ACCELERATION: The rate of change of velocity with respect to time,

$$a(t) = \frac{dv(t)}{dt} = \frac{d^2x(t)}{dt^2}$$

- **Average acceleration:** $a_{\text{avg}} = \frac{\Delta v}{\Delta t}$
- **Instantaneous acceleration:** The limit of a_{avg} as $\Delta t \rightarrow 0$

Motion with Constant Acceleration

When acceleration a is **constant**, we can derive the kinematic equations using integrals:

1. Velocity

$$a = \frac{dv}{dt} \implies dv = a \, dt$$

Integrating both sides from initial velocity v_0 to v , and from $t = 0$ to t :

$$\int_{v_0}^v dv = \int_0^t a \, dt \implies v - v_0 = a t \implies v(t) = v_0 + a t$$

2. Position

$$v(t) = \frac{dx}{dt} = v_0 + a t$$

Hence,

$$dx = (v_0 + a t) \, dt$$

Integrate from x_0 at $t = 0$ to x at time t :

$$\int_{x_0}^x dx = \int_0^t (v_0 + a \tau) d\tau \implies x - x_0 = v_0 t + \frac{1}{2} a t^2 \implies x(t) = x_0 + v_0 t + \frac{1}{2} a t^2$$

3. Velocity-Displacement Relation

Eliminating time t between $v = v_0 + a t$ and $x = x_0 + v_0 t + \frac{1}{2} a t^2$ gives another useful formula:

$$v^2 = v_0^2 + 2 a (x - x_0)$$

Example

Problem: An object starts from rest ($v_0 = 0$) at $x_0 = 0$, accelerating at 3 m/s^2 . Find its velocity and position at $t = 5 \text{ s}$.

Solution:

1. Since $v(t) = v_0 + a t = 0 + 3t$, at $t = 5 \text{ s}$:

$$v(5) = 3 \times 5 = 15 \text{ m/s}$$

2. Integrating for position, $x(t) = x_0 + v_0 t + \frac{1}{2} a t^2 = 0 + 0 + \frac{1}{2} (3)(5^2) = \frac{3}{2} \times 25 = 37.5 \text{ m}$
So, after 5 seconds, velocity is 15 m/s and position is 37.5 m from the starting point.

Final Summary & Takeaways

- **Displacement** measures change in position along a line.
- **Velocity** is the rate of change of position. Average velocity looks at a time interval, while instantaneous velocity is the derivative of position.
- **Acceleration** is the rate of change of velocity. Average acceleration uses Δv over Δt , while instantaneous acceleration is the derivative of velocity.
- **Constant acceleration** formulas can be derived via integrals:

$$v(t) = v_0 + a t, \quad x(t) = x_0 + v_0 t + \frac{1}{2} a t^2, \quad v^2 = v_0^2 + 2 a (x - x_0)$$

- These equations form the basis of 1D kinematics and are widely used to solve motion problems.

Raw Notes

✿ Raw Notes



5. Kinematics in 2D and 3D

From 1D to 2D/3D Motion

In higher dimensions, an object's position is represented by a vector rather than a single number. The same basic concepts apply as in 1D, but each coordinate is now a function of time.

POSITION VECTOR: A vector that describes the location of an object in space. In 2D, it is written as:

$$\mathbf{r}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}}$$

and in 3D as:

$$\mathbf{r}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}}$$

Units: meters (m).

Give the coordinates (x ; y ;z) of the vector you want to see constructed

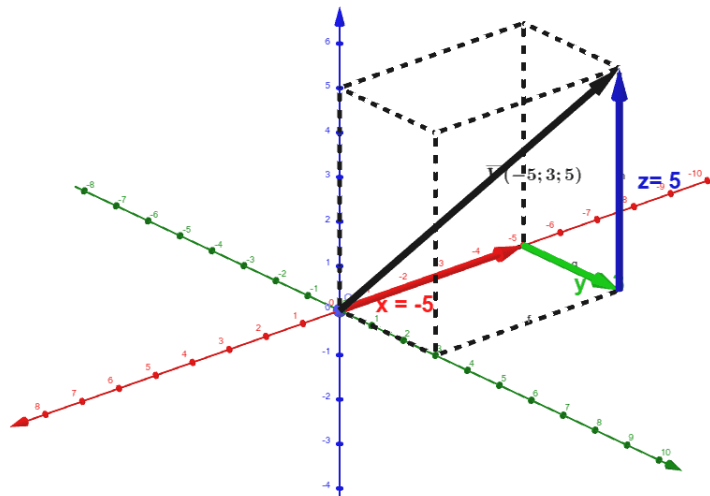
x=-5

y=3

z=5

See

Speed = 3.2

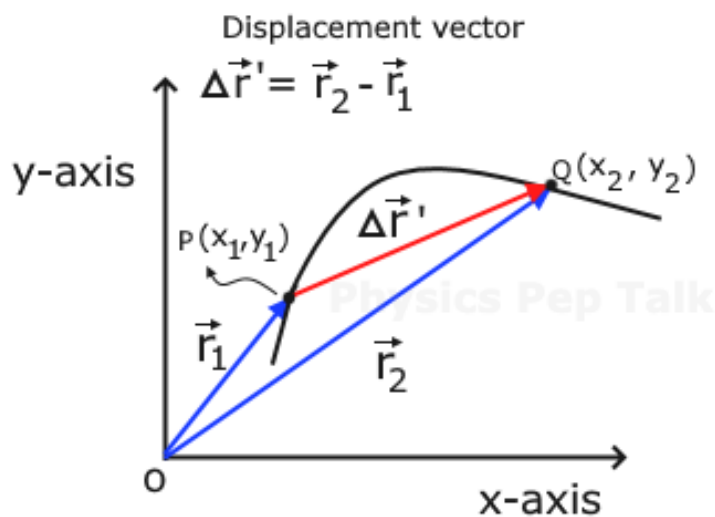



Displacement in 2D and 3D

Displacement is the change in the position vector over a time interval:

$$\Delta \mathbf{r} = \mathbf{r}(t_2) - \mathbf{r}(t_1)$$

- Note:* Displacement is a vector and accounts for both magnitude and direction.



Velocity

Average Velocity

Average velocity over a time interval $[t_1, t_2]$ is defined as:

$$\mathbf{v}_{\text{avg}} = \frac{\mathbf{r}(t_2) - \mathbf{r}(t_1)}{t_2 - t_1}$$

- *Units:* meters per second (m/s).

Instantaneous Velocity

Instantaneous velocity is the derivative of the position vector with respect to time:

$$\mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt} = \left(\frac{dx}{dt}\right)\hat{\mathbf{i}} + \left(\frac{dy}{dt}\right)\hat{\mathbf{j}} \quad (+ \left(\frac{dz}{dt}\right)\hat{\mathbf{k}} \text{ in 3D})$$

- Graphically, it is the slope of the tangent line to the x - t (or r - t) curve.

Acceleration

Acceleration is the rate of change of velocity with respect to time.

Average Acceleration

$$\mathbf{a}_{\text{avg}} = \frac{\mathbf{v}(t_2) - \mathbf{v}(t_1)}{t_2 - t_1}$$

Instantaneous Acceleration

$$\mathbf{a}(t) = \frac{d\mathbf{v}(t)}{dt} = \frac{d^2\mathbf{r}(t)}{dt^2} = \left(\frac{d^2x}{dt^2}\right)\hat{\mathbf{i}} + \left(\frac{d^2y}{dt^2}\right)\hat{\mathbf{j}} \quad (+ \left(\frac{d^2z}{dt^2}\right)\hat{\mathbf{k}} \text{ in 3D})$$

- *Units:* meters per second squared (m/s²).

Projectile Motion (2D Motion Example)

When an object is launched with an initial speed v_0 at an angle θ (relative to the horizontal), under uniform gravity g and no air resistance, we have:

- No acceleration in the x -direction ($a_x = 0$)
- A constant acceleration $-g$ in the y -direction

Equations of Motion

INITIAL VELOCITIES:

$$v_{0x} = v_0 \cos \theta, \quad v_{0y} = v_0 \sin \theta$$

POSITION:

$$\begin{aligned} x(t) &= v_{0x} t = v_0 \cos \theta \cdot t, \\ y(t) &= v_{0y} t - \frac{1}{2} g t^2 = v_0 \sin \theta \cdot t - \frac{1}{2} g t^2 \end{aligned}$$

VELOCITY:

$$\mathbf{v}(t) = \langle v_{0x}, v_{0y} - g t \rangle = \langle v_0 \cos \theta, v_0 \sin \theta - g t \rangle$$

The velocity vector remains tangent to the parabolic trajectory.

Projectile motion is a common example in two dimensions. When an object is projected, its motion can be decomposed into horizontal and vertical components.

- **Key Quantities:**

- Time of flight determined by the vertical motion.
- Horizontal range calculated from $x(t)$ at the time of landing.

Projectile Motion: An ideal example to analyze motion in 2D.

A projectile is a particle that is given an initial velocity and then follows a path determined entirely by the effects of gravitational acceleration.



In our idealized model we will:

- represent the projectile as a single particle
- assume constant gravitational acceleration $\vec{g}$
- assume no air resistance and other effects

Projectile motion is a motion in 2D. We will analyze the motion as a combination of horizontal motion with constant velocity, and vertical motion with constant acceleration.

$$a_x = 0$$

$$v_x = v_{0x}$$

$$x = x_0 + v_{0x}t$$

$$a_y = -g$$

$$v_y = v_{0y} - gt$$

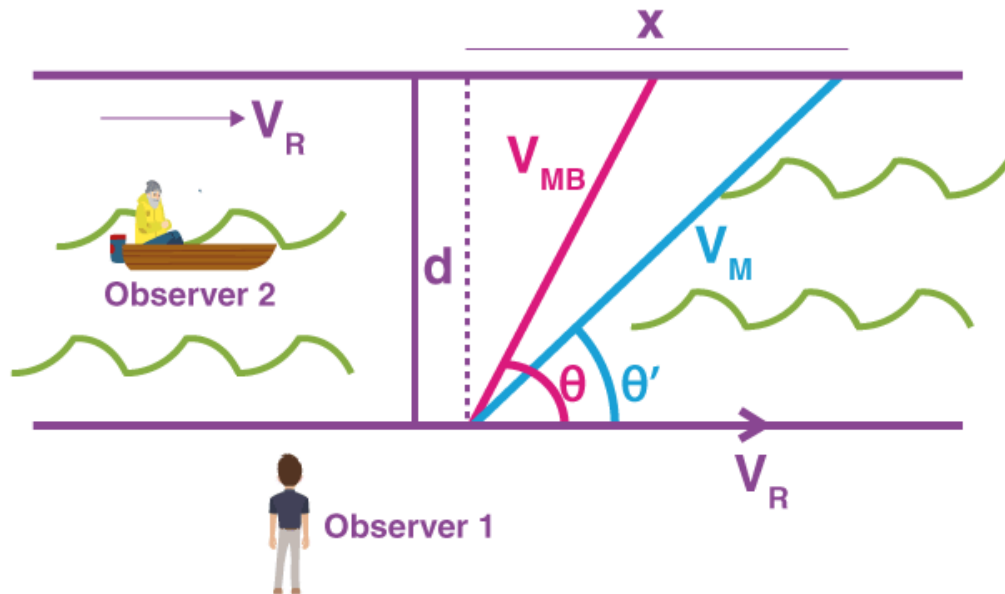
$$y = y_0 + v_{0y}t - \frac{1}{2}gt^2$$

Relative Velocity

Relative velocity describes the velocity of an object as observed from different frames of reference:

$$\mathbf{v}_{P/A} = \mathbf{v}_{P/B} + \mathbf{v}_{B/A}$$

- $\mathbf{v}_{P/A}$ is the velocity of particle P relative to observer A .
- $\mathbf{v}_{P/B}$ is the velocity of P relative to B .
- $\mathbf{v}_{B/A}$ is the velocity of B relative to A .



Example: Projectile from a Cliff

Consider a projectile launched horizontally from a cliff:

- **Initial Conditions:**

- Initial position: (x_0, y_0)
- Initial velocity: $\mathbf{v}_0 = (v_{0x}, 0)$

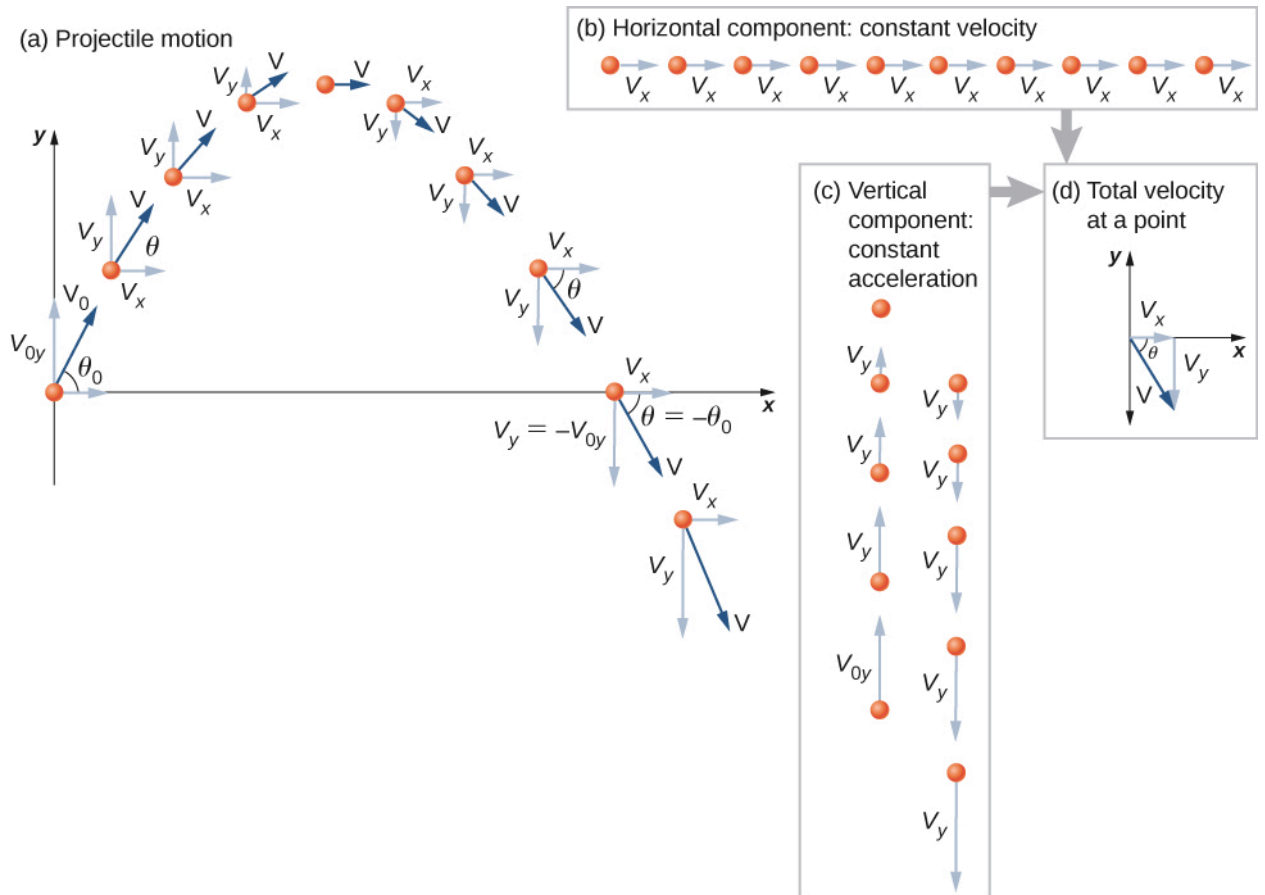
- **Equations of Motion:**

$$x(t) = x_0 + v_{0x}t$$

$$y(t) = y_0 - \frac{1}{2}gt^2$$

- **Analysis:**

- Time of flight is determined by when $y(t) = 0$ (assuming the ground is at $y = 0$).
- The horizontal range is then $x(t)$ at that time.



Deriving Maximum Height and Horizontal Range

Assume the projectile is launched from $y = 0$. We use the **integral or derivative** approaches as follows:

1. Maximum Height, $h_{\max}$

This occurs when vertical velocity becomes zero. From $v_y(t) = v_0 \sin \theta - g t$:

$$0 = v_0 \sin \theta - g t_h \quad \implies \quad t_h = \frac{v_0 \sin \theta}{g}$$

Substituting t_h into $y(t)$:

$$h_{\max} = y(t_h) = v_0 \sin \theta \left(\frac{v_0 \sin \theta}{g} \right) - \frac{1}{2} g \left(\frac{v_0 \sin \theta}{g} \right)^2$$

Simplify:

$$h_{\max} = \frac{v_0^2 \sin^2 \theta}{g} - \frac{v_0^2 \sin^2 \theta}{2g} = \frac{v_0^2 \sin^2 \theta}{2g}$$

If the projectile is not launched vertically ($\theta \neq \frac{\pi}{2}$), the formula remains valid; the maximum height just depends on the $\sin^2 \theta$ factor.

2. **Horizontal Range, $R_{\max}$**

The range is the horizontal distance when the projectile returns to $y = 0$. Solve $y(t) = 0$ for $t \neq 0$:

$$0 = v_0 \sin \theta t - \frac{1}{2} g t^2 \quad \implies \quad t(v_0 \sin \theta - \frac{1}{2} g t) = 0$$

The non-zero solution is

$$t_f = \frac{2v_0 \sin \theta}{g}$$

Plug into $x(t)$ to get

$$R_{\max} = x(t_f) = v_0 \cos \theta \cdot \left(\frac{2v_0 \sin \theta}{g} \right) = \frac{2v_0^2 \sin \theta \cos \theta}{g}$$

Using the identity $\sin(2\theta) = 2 \sin \theta \cos \theta$:

$$R_{\max} = \frac{v_0^2 \sin(2\theta)}{g}$$

Constant Acceleration (Kinematics) Cheat Sheet

Velocity as a Function of Time:

$$v = v_0 + a t$$

Position as a Function of Time:

$$x = x_0 + v_0 t + \frac{1}{2} a t^2$$

Velocity as a Function of Position (Time Eliminated):

$$v^2 = v_0^2 + 2 a (x - x_0)$$

Displacement Using Average Velocity:

The average velocity under constant acceleration is

$$\bar{v} = \frac{v+v_0}{2}$$

so that the displacement is given by

$$x - x_0 = \bar{v} t = \frac{1}{2}(v + v_0) t$$

Summary

- **Position and Displacement:** In 2D/3D, position is described by a vector $\mathbf{r}(t)$, and displacement is the difference between two position vectors.
- **Velocity:** Average velocity is the displacement divided by the time interval, while instantaneous velocity is the derivative $\mathbf{v}(t) = \frac{d\mathbf{r}}{dt}$.
- **Acceleration:** Defined as the derivative of velocity, $\mathbf{a}(t) = \frac{d\mathbf{v}}{dt}$, with both average and instantaneous forms.
- **Projectile Motion:** Motion in two dimensions where horizontal and vertical motions are treated independently.
- **Relative Velocity:** How velocities transform between different reference frames.

Understanding these vector-based kinematic concepts is crucial for analyzing complex motions in both two and three dimensions.

Self Test

✿ [Self-Test: Lecture 5](#)

Raw Notes

✿ [Raw Notes](#)



6. Kinematics in 2D and 3D - Circular Motion, Forces, and Newton's Laws

Circular Motion

Circular Motion

(Cir-cu-lar-Mo-tion)

Definition:

Circular motion is the movement of an object along the circumference of a circle or a circular path.

Formula: $F_c = mv^2/r$

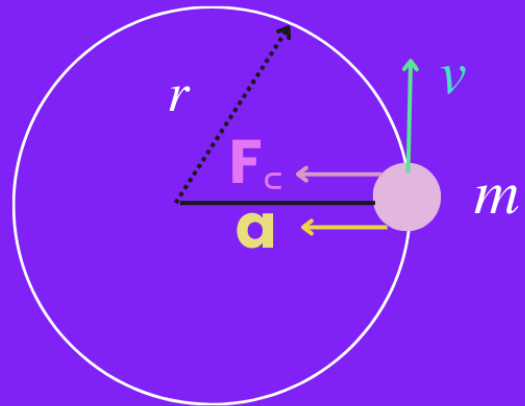
Where:

F_c = Centripetal Force (N)

m = Mass of the object (kg)

v = Tangential velocity (m/s)

r = Radius of the circular path (m)



Ex Examples.com

Uniform Circular Motion (UCM)

- **Position Function:**

The position vector for an object moving in a circle of radius R is given by:

$$\mathbf{r}(t) = R \cos(\theta(t)) \mathbf{i} + R \sin(\theta(t)) \mathbf{j}$$

- **Tangential Displacement and Velocity:**

- **Displacement:**

In circular motion, the small displacement Δs along the circle is tangential to the path. For a small angle θ (in radians), the arc length is:

$$\Delta s = R\theta$$

- **Velocity:**

The velocity vector is the time derivative of the position vector and is always tangent to the circular path:

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} \quad (\text{tangent to the circle})$$

Even though the speed (magnitude of velocity) might be constant in UCM, the direction changes continuously.

- **Acceleration (Radial/Normal Component):**

For uniform circular motion, the radial (centripetal) acceleration, which is directed inward, is:

$$a_{\text{rad}} = \frac{v^2}{R}$$

Using a small-angle approximation ($\sin \theta \approx \theta$), one can relate the changes in displacement and velocity:

$$\Delta s = v\Delta t \quad \text{and} \quad \Delta v = v\theta$$

Taking the limit as $\Delta t \rightarrow 0$ leads to the acceleration formula.

Uniform Circular Motion

Uniform circular motion occurs when an object moves in a circular path with constant speed.

$$a_c = \frac{v^2}{r}$$

- v is the tangential speed of the object,
- r is the radius of the circular path.

Ex [Examples.com](https://www.examples.com)



Non-Uniform Circular Motion

- **Characteristics:**

In non-uniform circular motion, the speed is not constant ($|\mathbf{v}_1| \neq |\mathbf{v}_2|$).

- **Tangential Acceleration:**

There is an additional component of acceleration along the tangent to the circle due to the change in speed:

$$a_{\text{tan}} = \frac{dv}{dt}$$

- **Total Acceleration:**

The total acceleration is the vector sum of the radial (centripetal) and tangential accelerations:

$$\mathbf{a} = \mathbf{a}_{\text{rad}} + \mathbf{a}_{\text{tan}}$$

with the radial acceleration still given by:

$$a_{\text{rad}} = \frac{v^2}{R}$$

and directed inward.

Non-Uniform Circular Motion

Tangential Acceleration

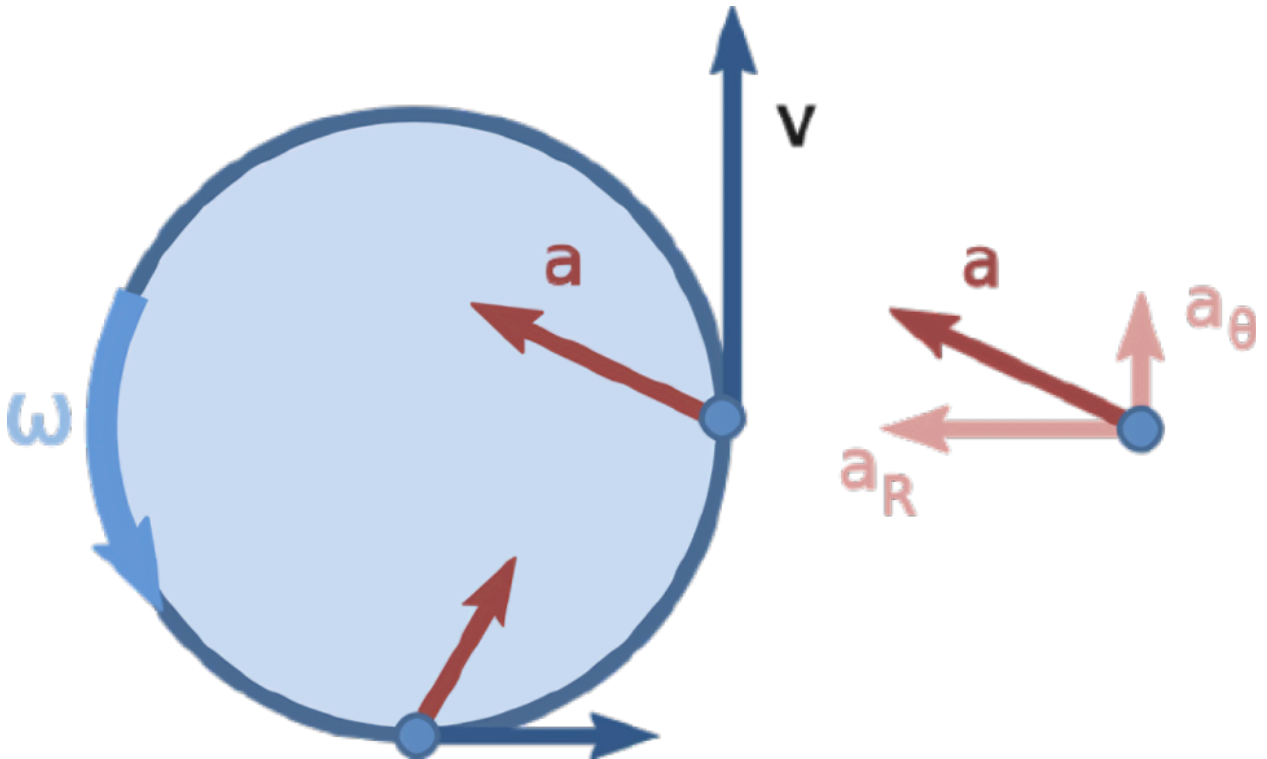
$$a_t = \frac{d|v|}{dt}$$

The component of acceleration that is tangential to the circular path, responsible for changing the speed of the object in non-uniform circular motion.

Total Acceleration

$$a = \sqrt{a_x^2 + a_t^2}$$

In non-uniform circular motion, the total acceleration a is the vector sum of the centripetal acceleration a_x and the tangential acceleration



Period and Frequency

- **Period (T):**

The time required for one complete cycle (full circle). For uniform circular motion:

$$T = \frac{2\pi R}{v} \quad (\text{seconds})$$

- **Frequency (f):**

The number of cycles per unit time:

$$f = \frac{1}{T} \quad (\text{Hz})$$

- **Radial Acceleration in Terms of Period:**

Expressing a_{rad} using the period:

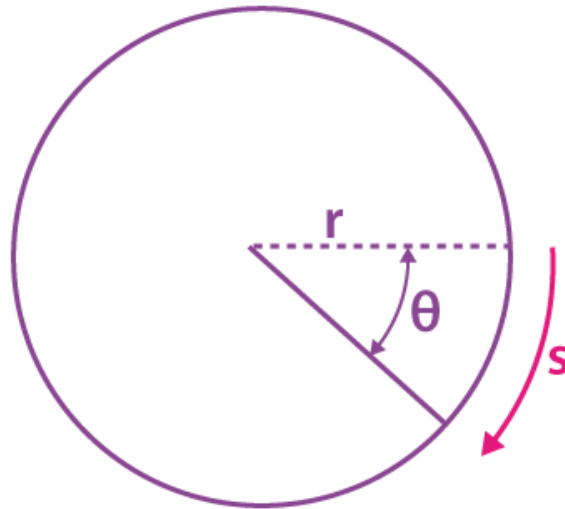
$$a_{\text{rad}} = \frac{4\pi^2 R}{T^2}$$

Angular vs. Linear Displacement and Velocity

- **Angular Displacement (θ):**

Measured in radians, it represents the angle through which an object has rotated.

Angular displacement



- **Linear Displacement (s):**

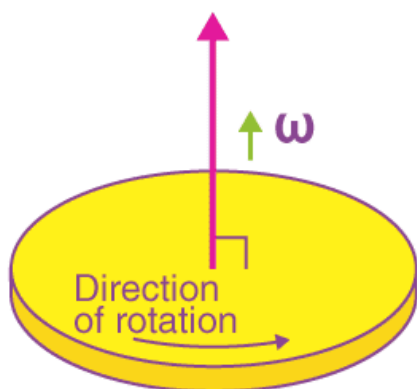
The arc length on the circle corresponding to the angular displacement:

$$s = R\theta$$

- **Angular Velocity (ω):**

The rate of change of angular displacement:

$$\omega = \frac{d\theta}{dt}$$



- **Linear Velocity (v):**

Related to angular velocity by:

$$v = \omega R$$

This shows the direct relation between the angular motion and the linear speed along the circular path.

Forces

Overview of Fundamental Forces

- **Gravitational Force:**

The attractive force between two masses, described by Newton's law of universal gravitation.

- **Electromagnetic Force:**

The force between charged particles, responsible for electricity, magnetism, and light.

- **Strong Nuclear Force:**

The force that holds the protons and neutrons together in the nucleus.

- **Weak Nuclear Force:**

Responsible for certain types of radioactive decay.

- **Fifth Force (Hypothetical):**

Some theories suggest the existence of an additional force beyond the four known fundamental forces. Experiments such as the Eötvös experiment have investigated potential anomalies.

Force as a Vector

- **Definition:**

Force is a vector quantity, meaning it has both magnitude and direction.

- **Components:**

Any force vector can be broken down into components (e.g., horizontal and vertical).

- **Vector Addition:**

The net force acting on an object is the vector sum of all individual forces acting on it:

$$\mathbf{F}_{\text{net}} = \sum \mathbf{F}_i$$

Newton's Laws of Motion

Newton's First Law (Law of Inertia)

- **Statement:**

An object at rest remains at rest, and an object in motion continues in motion with a constant velocity unless acted upon by a net external force.

- **Inertial Frame of Reference:**

Newton's laws are valid in inertial frames—reference frames that are not accelerating. In a non-inertial frame, additional fictitious forces (like the centrifugal force) appear.

- **Examples:**

- A hockey puck sliding on ice eventually slows due to friction (an external force).
- In space (an inertial environment), an astronaut will float indefinitely if no forces act.

Newton's Second Law (Law of Acceleration)

- **Statement:**

The net force acting on an object is equal to the mass of the object multiplied by its acceleration:

$$\mathbf{F} = m\mathbf{a}$$

- **Explanation:**

This law quantitatively relates the change in an object's motion to the applied force.

- **Examples:**

- Pushing a shopping cart: A greater force produces a higher acceleration.
- Circular motion: The centripetal force required to maintain circular motion is given by $F = m\frac{v^2}{R}$.

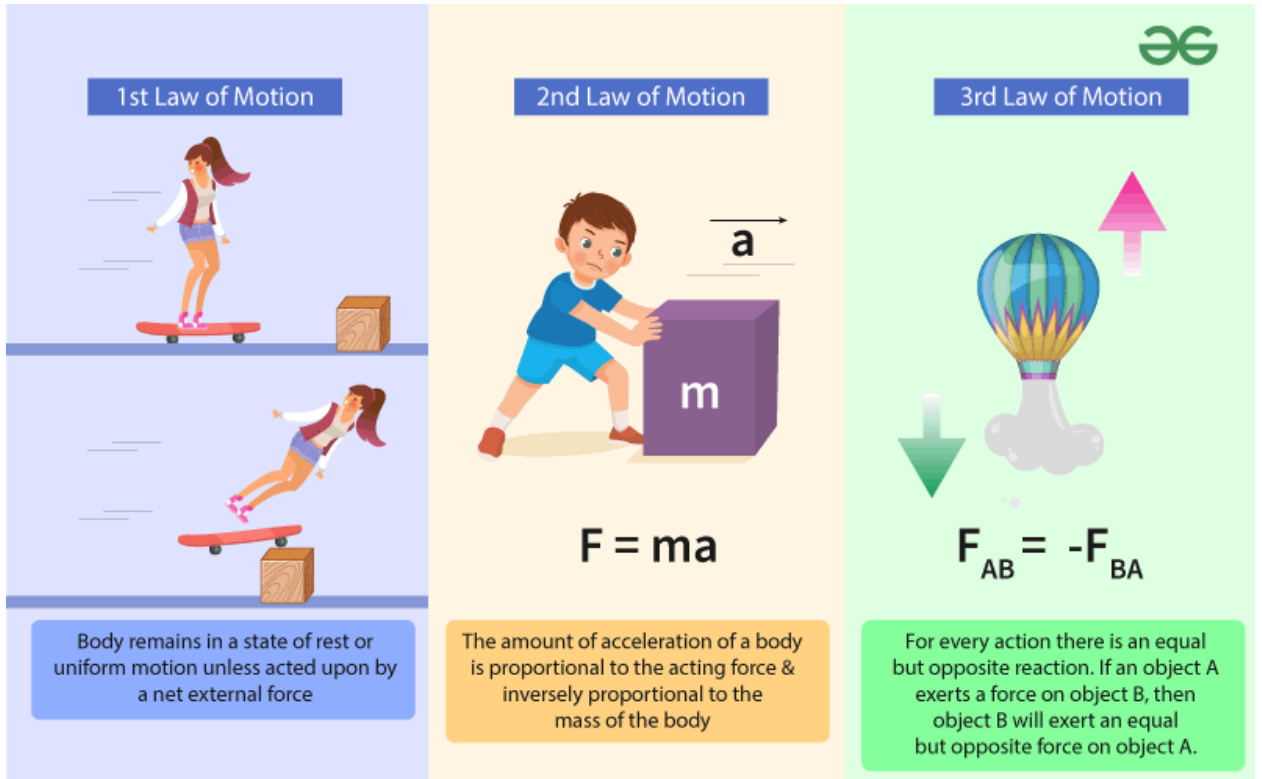
Newton's Third Law (Action-Reaction Law)

- **Statement:**

For every action, there is an equal and opposite reaction. If object A exerts a force on object B, then object B exerts a force of equal magnitude but opposite direction on object A.

- **Examples:**

- Rocket propulsion: Expelling gas out of the rocket's engine produces a thrust in the opposite direction.
- Walking: The foot pushes backward against the ground, and the ground pushes the foot forward.



Mass vs. Weight

- **Mass:**

A scalar quantity that measures the amount of matter in an object (measured in kilograms).

- **Weight:**

A vector quantity that represents the gravitational force acting on an object:

$$\text{Weight} = m \cdot g$$

where g is the acceleration due to gravity (approximately 9.8 m/s^2 on Earth).

- **Key Difference:**

Mass is constant regardless of location, while weight varies with the local gravitational field.

Final Summary & Key Takeaways

- **Circular Motion:**

- In uniform circular motion, displacement and velocity are tangential, with constant speed but continuously changing direction.
- Non-uniform circular motion introduces tangential acceleration in addition to centripetal (radial) acceleration.
- The relationship between angular and linear quantities (θ vs. s , ω vs. v) is crucial in analyzing circular motion.
- **Forces:**
 - Fundamental forces (gravitational, electromagnetic, strong, weak, and possibly a fifth force) govern the interactions between objects.
 - Force is a vector and is the basis for understanding motion through vector addition.
- **Newton's Laws:**
 - Newton's First Law emphasizes inertia and the importance of inertial frames.
 - The Second Law quantifies the relation between force, mass, and acceleration.
 - The Third Law establishes the concept of action-reaction pairs.
 - The distinction between mass and weight is fundamental in physics.

This note integrates the provided details with additional context and explanations to create a clear, comprehensive resource on circular motion, forces, and Newton's laws.

Raw Notes

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7. Newton's Laws - Force Analysis

Gravitational Force

GRAVITATIONAL FORCE: Gravitational force, or weight, is the force exerted by the Earth to pull objects toward its center.

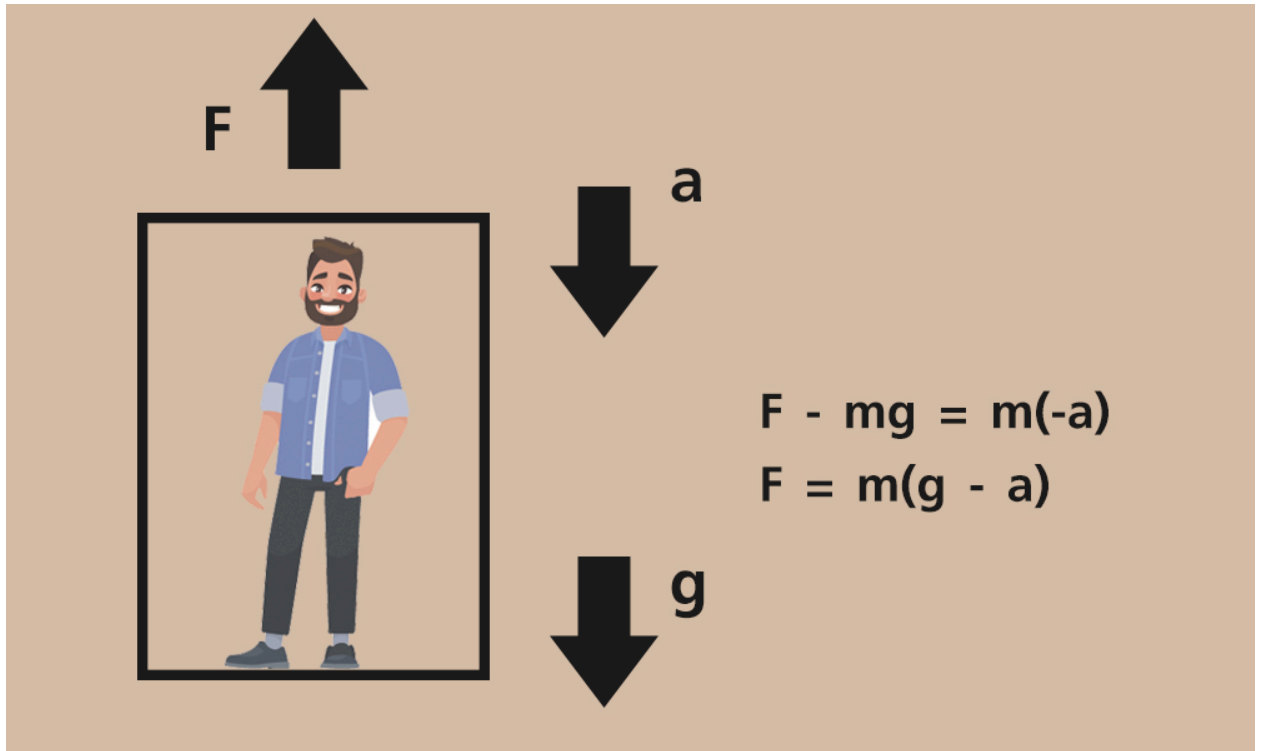
- **Formula:**

$$F_g = m \times g$$

- m : Mass of the object
- g : Acceleration due to gravity (approximately 9.8 m/s^2).

- **Key Points:**

- Acts vertically downward.
- Independent of the object's orientation.



Normal Force

NORMAL FORCE: The normal force is the support force provided by a surface, acting perpendicular to that surface.

- **Characteristics:**

- On a flat surface, the normal force equals the gravitational force in magnitude.
- On an inclined surface, the normal force is reduced and calculated as:

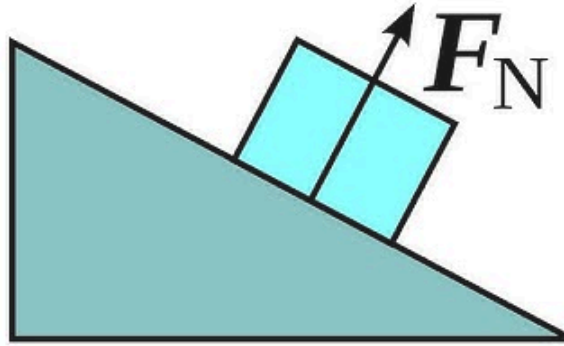
$$F_N = m \times g \times \cos(\theta)$$

where θ is the angle of the incline.

- **Usage:**

Adjusts to balance forces perpendicular to the contact surface.

Normal Force

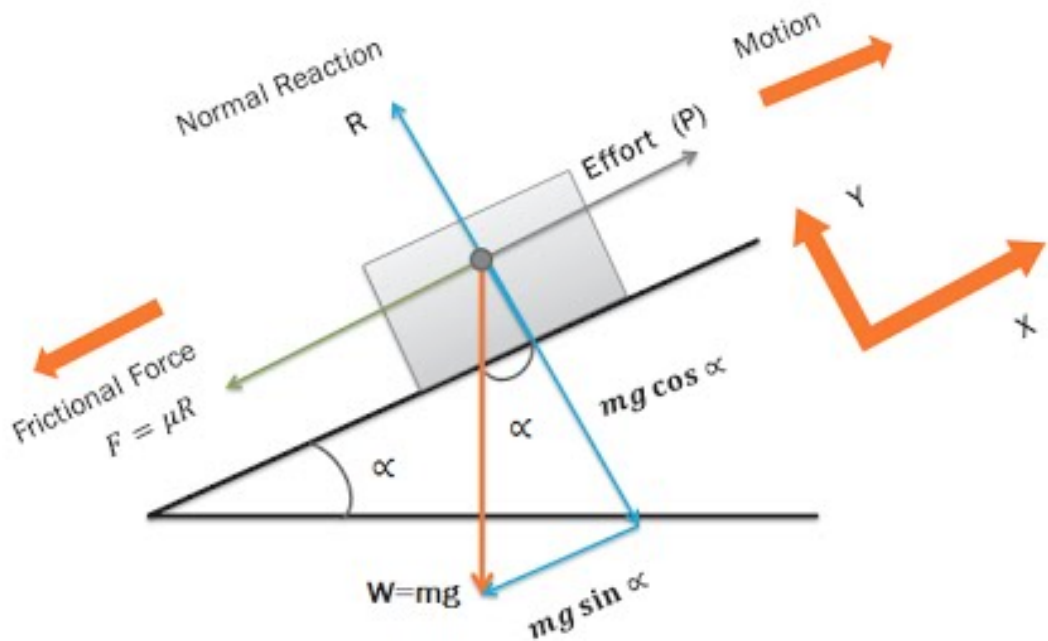


Force Analysis & Free Body Diagram (FBD)

A Free Body Diagram is an essential tool to visualize all external forces acting on an object. When drawing an FBD:

- **Steps:**
 - **Isolate the Object:** Represent it as a simple shape or a dot.
 - **Identify All Forces:**
 - **Gravitational Force** (F_g) : Drawn as an arrow pointing straight down.
 - **Normal Force** (F_N) : Drawn perpendicular to the contact surface.
 - **Applied or Other Forces:**
 - If a force is neither horizontal nor vertical, show its angle relative to a reference axis (horizontal or vertical).
 - Clearly label the force and its angle.

Remark: Accurate angle representation is crucial when forces act at an angle to ensure proper component resolution.



Applying Newton's Laws

Newton's laws are the foundation of force analysis and motion prediction.

Newton's First Law (Law of Inertia)

Statement: An object remains at rest or moves with a constant velocity unless acted upon by a net external force.

- **Formula:**

$$\sum F = 0 \Rightarrow a = 0$$

- **Application:**

If all forces balance (net force equals zero), the object remains in equilibrium.

Newton's Second Law (Law of Acceleration)

Statement: The net force acting on an object equals its mass multiplied by its acceleration.

- **Formula:**

$$\sum F = m \times a$$

- **Application Examples:**

- **Flat Surface:**

$$F_N - F_g = 0 \quad \Rightarrow \quad F_N = m \times g$$

- **Inclined Plane:**

Resolve forces into components parallel and perpendicular to the incline to determine the net force and acceleration.

Newton's Third Law (Action-Reaction)

Statement: For every action, there is an equal and opposite reaction.

- **Formula:**

$$F_{A \rightarrow B} = -F_{B \rightarrow A}$$

- **Example:**

The normal force exerted on an object by a surface is met by an equal force exerted by the object on that surface.

Conclusion

Understanding how to analyze forces and apply Newton's laws is crucial for solving problems in mechanics. By:

- Identifying gravitational and normal forces,
- Constructing a clear Free Body Diagram with proper force directions (including angles for non-vertical/horizontal forces),
- And applying Newton's laws, you can accurately predict and analyze an object's motion in various physical scenarios.

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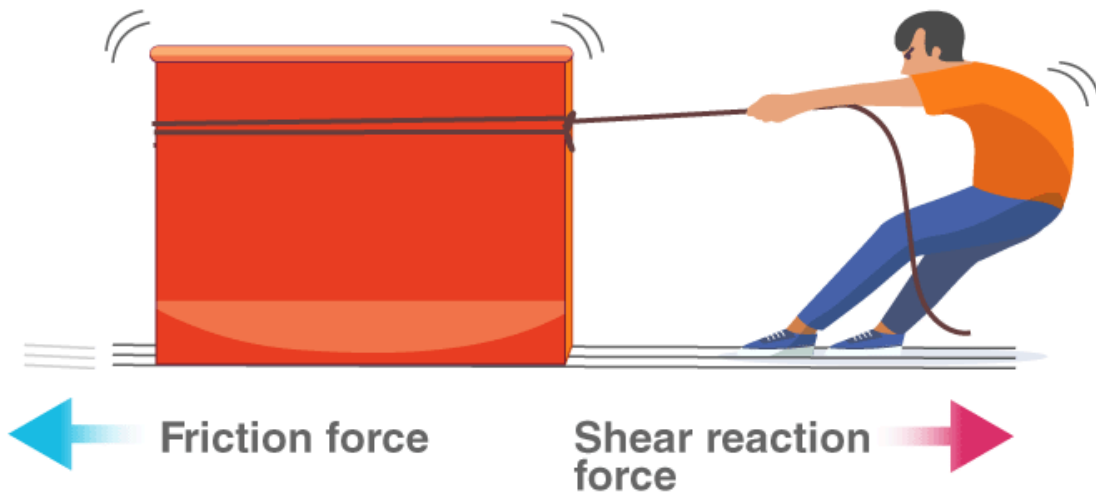
8. Friction and Its Role in Multi-Body Systems

This note explains friction, its types, and how friction forces interact between objects in contact. It also covers the role of friction in uniform circular motion where there is no tangential acceleration.

Friction

FRICTION: Friction is the force that acts parallel to the contact area between two surfaces. Its direction is opposite to the relative motion between the surfaces. When one object is on top of another, friction can either prevent slipping or allow the objects to move together if the frictional force is sufficient.

- Friction always acts along the interface between surfaces.
- According to Newton's Third Law, the friction force acting on one object has an equal and opposite counterpart on the other.
- Friction exists even in the absence of motion (static friction).



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Kinetic Friction

KINETIC FRICTION: Kinetic friction occurs when two surfaces slide relative to each other. It is calculated by:

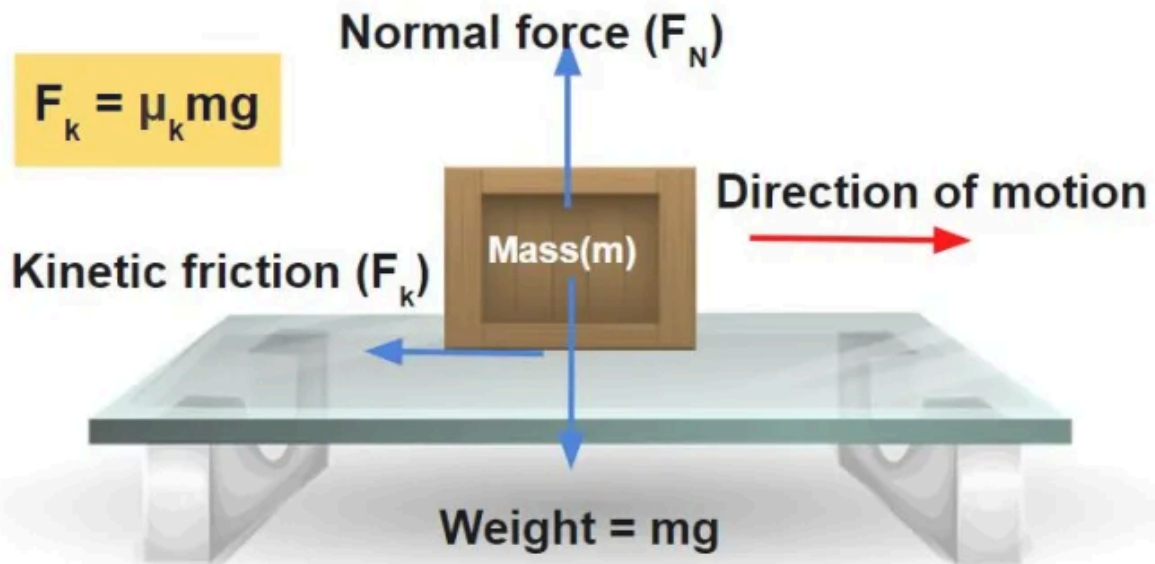
$$f_k = \mu_k \times N$$

where μ_k is the coefficient of kinetic friction and N is the normal force.

- Kinetic friction opposes the relative sliding motion between surfaces.

Formula for Kinetic Friction

DewWool



Static Friction

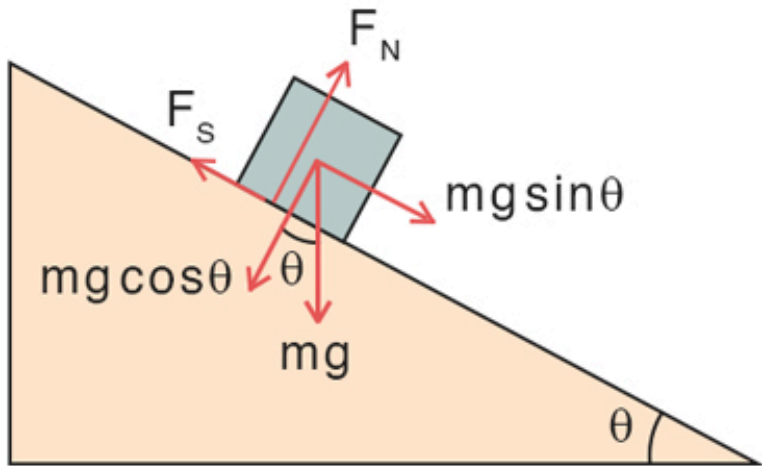
STATIC FRICTION: Static friction acts when there is no relative motion between two contacting surfaces. Its maximum value is:

$$f_s \leq \mu_s \times N$$

where μ_s is the coefficient of static friction (typically greater than μ_k) and N is the normal force.

- Static friction adjusts to balance an applied force up to its maximum limit.
- When the applied force F is less than or equal to this maximum, the objects remain stationary.
- If F exceeds the maximum static friction, motion begins and kinetic friction takes over.

Static Friction on an Inclined Plane



$$\mu_s = \tan \theta$$

F_N : Normal force

F_s : Static friction

m : Mass of the block

θ : Angle of inclination

μ_s : Coefficient of static friction g : Acceleration due to gravity

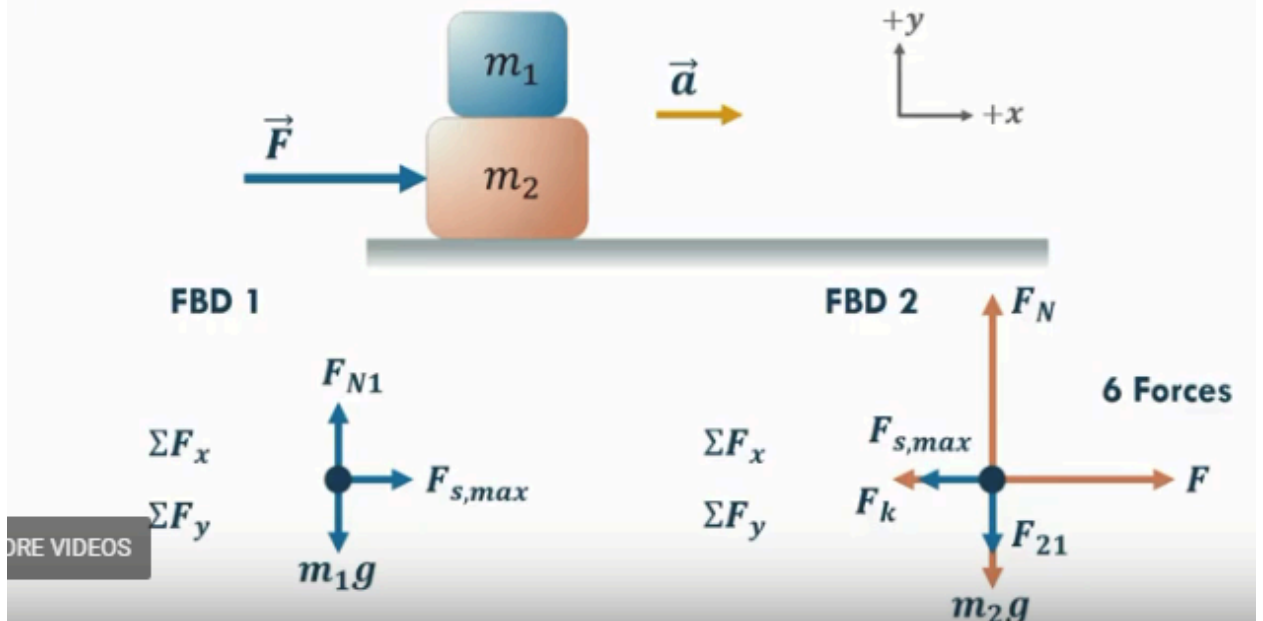


Friction in Multi-Body Systems

- When one object is placed on top of another, friction determines whether the objects move together or slip relative to each other.
- In a system where m_2 is on top of m_1 and $m_1 = 2m_2$:
 - The static friction between the two objects can allow them to move as a single unit if it is strong enough.
 - By Newton's Third Law, the friction force exerted on m_2 by m_1 is equal in magnitude and opposite in direction to the friction force exerted on m_1 by m_2 .

Stacked Blocks

E

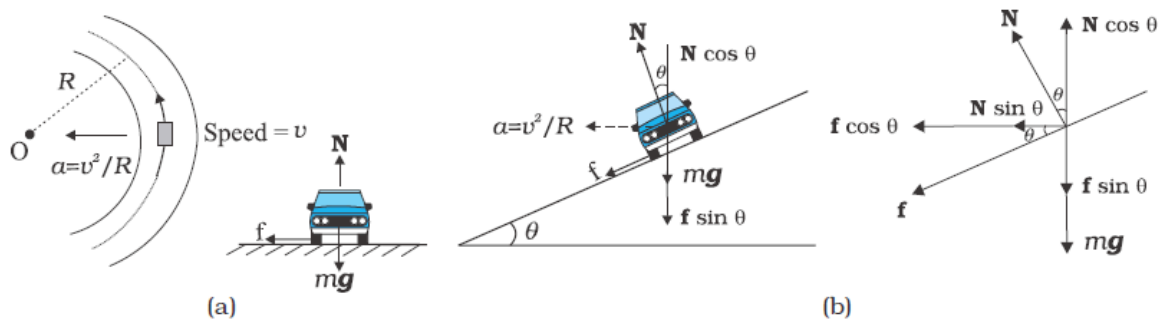


Friction in Uniform Circular Motion

- In uniform circular motion, there is no tangential acceleration ($a_{tan} = 0$); the net force is entirely radial (centripetal).
- The net force needed to maintain circular motion is:

$$F_{net} = m \times a_{rad}$$

- Friction (or another force) may act as the centripetal force, directing the object towards the center of the circle.



Circular motion of a car on (a) a level road, (b) a banked road.

Conclusion

Understanding friction is essential in analyzing the dynamics of multi-body systems. Friction, acting parallel to surfaces, can either prevent or facilitate relative motion. In systems with objects in contact, such as when one object rests on another, friction may allow them to accelerate together if it is sufficient. Additionally, in uniform circular motion, friction can provide the necessary centripetal force even when no tangential acceleration is present.

Raw Notes

✿ Raw Notes

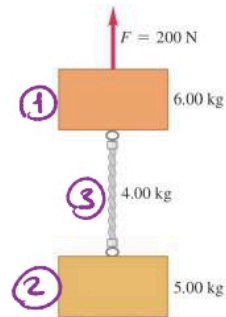


9. Midterm 1 Preparation

PHYS101 - Fall '23 - PS4 - Ch4 (~5) ~ Baran Arslan

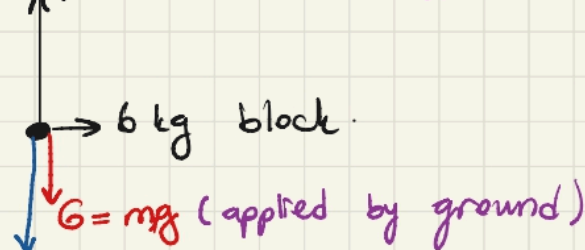
4.46 •• The two blocks in Fig. P4.46 are connected by a heavy uniform rope with a mass of 4.00 kg. An upward force of 200 N is applied as shown. (a) Draw three free-body diagrams: one for the 6.00 kg block, one for the 4.00 kg rope, and another one for the 5.00 kg block. For each force, indicate what object exerts that force. (b) What is the acceleration of the system? (c) What is the tension at the top of the heavy rope? (d) What is the tension at the midpoint of the rope?

Figure P4.46



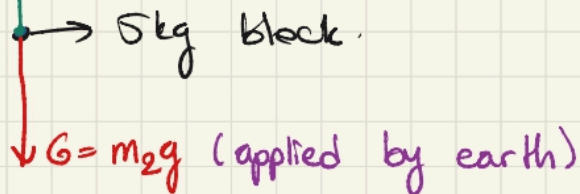
part a) Arrows are not to scale:

1) $F = 200 \text{ N}$ (external/applied force)

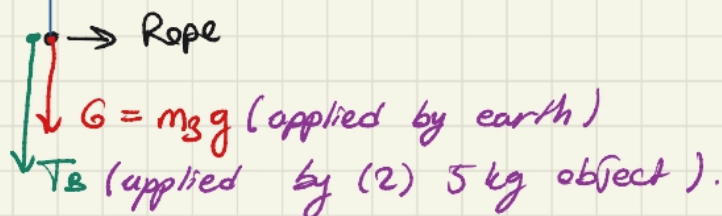


T_u : Tension of the rope at the upper end of the rope (applied by rod)

2) T_b : Tension of the rope at the bottom of the rope (applied by rod)

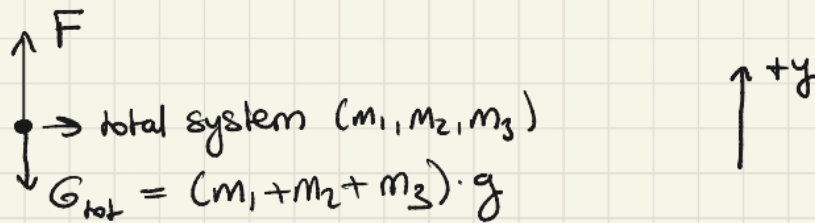


3) T_u (applied by (1) 6 kg block)



part b

If we consider the system as a whole then we got the following free body diagram.



Then $\vec{F}_{tot} = \vec{F} - \vec{G}_{tot} = m_{tot} \vec{a}$

$$\hookrightarrow \vec{a} = \frac{\vec{F} - \vec{G}_{tot}}{m_{tot}} = \frac{(200 - 15g)}{15} \approx \underline{\underline{3.5 \hat{j} \text{ m/s}^2}}$$

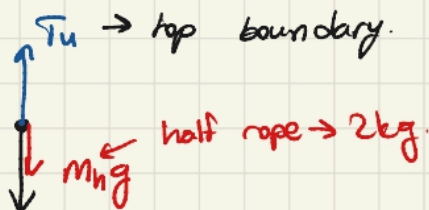
part c $\rightarrow$ Motion in 1D, $+\rightarrow +\hat{j}$ direction
 $-\rightarrow -\hat{j}$ direction.

$$m_1 a = F_{tot,1} = F - mg - T_u$$

$$\hookrightarrow T_u = F - mg - m_1 a = 200 - 6 \cdot (g + a) \approx \underline{\underline{120 \text{ N}}}$$

part d

We may take the top half of the rope as a system



T_m : Tension at the mid point of the rope } Just assume downwards if it is upwards, we simply get a $(-)$ sign.

$$m_h a = F_{\text{rot}h} = T_u - m_h g - T_m$$

$$\hookrightarrow T_m = m_h(a + g) - T_u \approx \underline{\underline{93 \text{ N}}}.$$

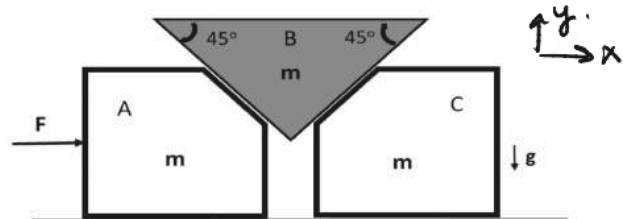
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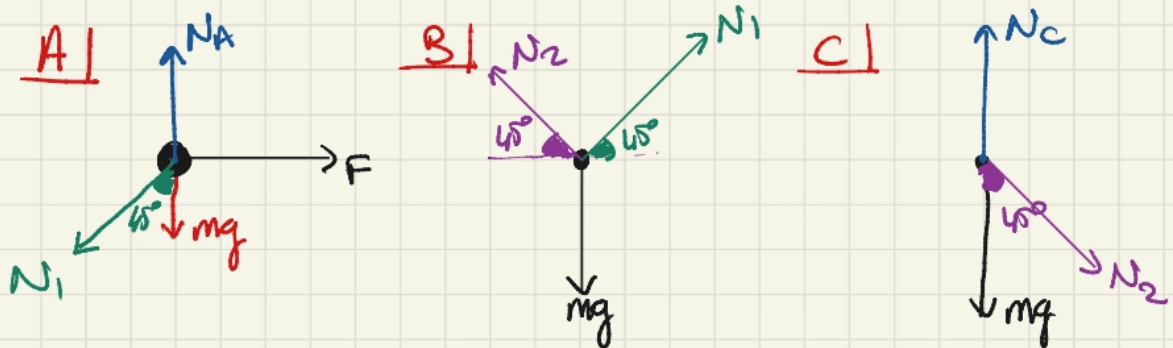
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Q3-(25 pts) Three blocks, labeled A, B & C as shown, all of them of mass m , are initially positioned as in the figure. There is a horizontal force of magnitude F on the leftmost block, and there is no friction on any of the surfaces in the system.

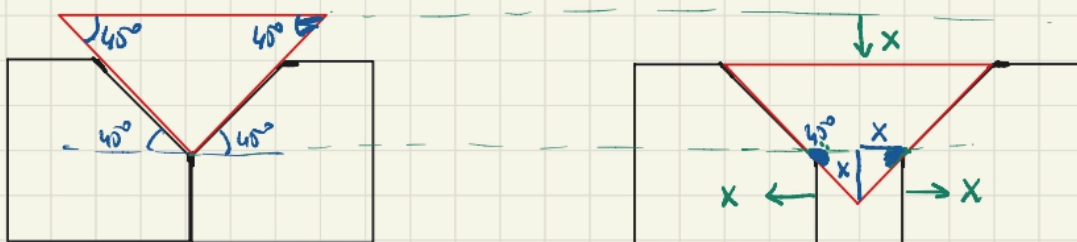


a) Draw the free body diagrams for all blocks.



b) If $F = 0$, find the acceleration of the triangular Block B.

Let us observe this system more closely



So, our system has some kind of an symmetry.

1) Displacement of A to the left is always equal to the disp. of B in vertical axis.

2) Displacement of C to the right is always equal to the disp. of B in vertical axis.

3) Therefore $|\text{disp. A}| = |\text{disp C}|$

or also $-a = a_{By} = a_{Ax} = -a_{Cx} \rightarrow \text{Discuss.}$
towards $-\hat{y}$ $\curvearrowright$ (**) (*)

And we can see from part a:

$$a_{Ax} = \frac{F - N_1 \cdot \sin 45^\circ}{m}$$

$$a_{Cx} = \frac{N_2 \cdot \sin 45^\circ}{m}$$

$$a_{By} = \frac{(N_1 + N_2) \cdot \sin 45^\circ - mg}{m}$$

If we use (*) and let $F=0$.

$$\frac{-N_1 \cdot \sin 45^\circ}{m} = \frac{-N_2 \cdot \sin 45^\circ}{m} \rightarrow N_1 = N_2 = N.$$

Then use (**)

$$\frac{-N \cdot \sin 45^\circ}{m} = \frac{2N \cdot \sin 45^\circ}{m} - g \rightarrow N = \frac{mg}{3 \cdot \sin 45^\circ}$$

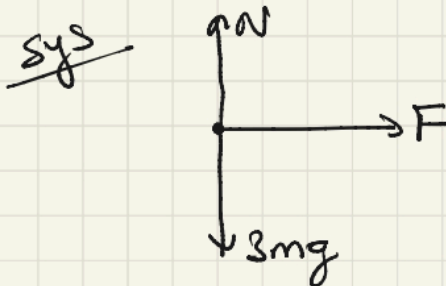
Then,

$$a_{By} = \frac{\frac{2mg}{3 \sin 45^\circ} \cdot \sin 45^\circ - mg}{m} = \boxed{-\frac{g}{3}}$$

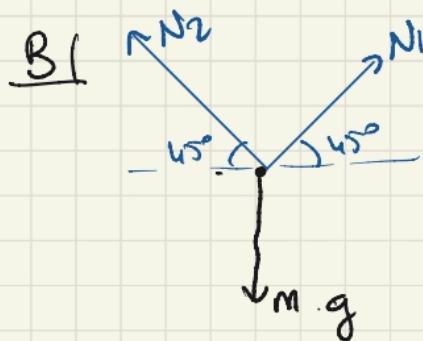
c) Determine the force F for which the vertical acceleration of the triangular Block B is zero?

→ As B doesn't have vertical acceleration, we can imagine this three object as a system moving together.

Let's observe this system



$$F_{\text{tot sys}} = F = 3m \cdot a_x \quad a_x = \frac{F}{3m}$$



$$(N_1 + N_2) \cdot \sin 45^\circ = mg \quad (1)$$

$$\rightarrow a_x \quad N_1 \cdot \cos 45^\circ - N_2 \cdot \cos 45^\circ = m a_x \quad (2)$$

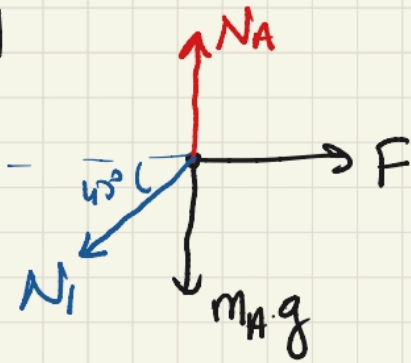
$$= \frac{F}{3}$$

$$(N_1 - N_2) \cdot \cos 45^\circ = \frac{m \cdot F}{3m} \quad (2)$$

$$(N_1 + N_2) \cdot \sin 45^\circ = mg \quad (1)$$

$$+ \quad 2N_1 \cos 45^\circ = \frac{F}{3} + mg \quad (*)$$

A



$$F_{\text{tot Ax}} = F - N_1 \cdot \cos 45^\circ = \underline{\underline{\max \frac{F}{3}}}$$

$$\underline{N_1 \cdot \cos 45^\circ} = F - \frac{F}{3} = \underline{\underline{\frac{2F}{3}}}$$

$$(*) \quad 2N_1 \cdot \cos 45^\circ = \frac{4F}{3} = \frac{F}{3} + mg$$

$$\hookrightarrow \boxed{F = mg}$$

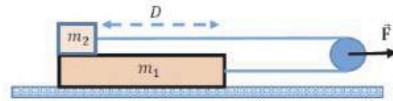
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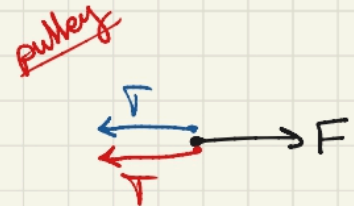
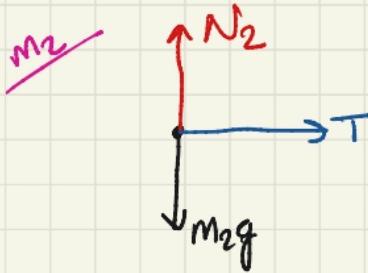
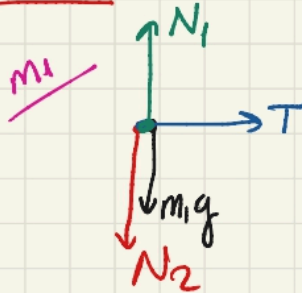
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Q2-(25 pts) A mass m_2 is on top of a platform with mass m_1 (assume $m_2 < m_1$). There is no friction between the mass and the platform, or between the platform and the horizontal floor. The two masses are connected by an ideal massless string going over a massless pulley. Initially m_2 is a distance D away from the right edge of the platform and all masses are at rest. Starting at time $t=0$, a force F is applied to the pulley horizontally, as shown in the figure.



a) Draw free-body diagrams of the blocks and the pulley for $t > 0$. (6pts)

part a



b) What is the tension on the string? (4 pts)

$$F - 2T = \underbrace{m_{\text{pulley}} \cdot a_{\text{pulley}}}_0 = 0$$

$$\hookrightarrow F = 2T \rightarrow T = F/2$$

$$T = F/2$$

c) Find how long it would take for the mass m_2 to cover the distance D to reach the edge of the platform. (8 pts)

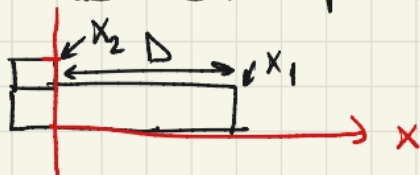


$$m_1 \cdot a_1 = T \rightarrow a_1 = \frac{T}{m_1}$$

$$m_2 \cdot a_2 = T \rightarrow a_2 = \frac{T}{m_2}$$

$$\left. \begin{array}{l} a_1 = \frac{T}{m_1} \\ a_2 = \frac{T}{m_2} \end{array} \right\} a_2 > a_1 \text{ since } m_1 > m_2$$

Let us set up a coord. sys. :



$x_1(t)$: position of right end of mass m_1 at time t .

$x_2(t)$: position of right end of mass m_2 at time t .

$$\hookrightarrow \left. \begin{aligned} x_1(t) &= D + \frac{a_1 t^2}{2} \\ x_2(t) &= \frac{a_2 t^2}{2} \end{aligned} \right\} \frac{a_2 t^2}{2} = D + \frac{a_1 t^2}{2}$$

$$(a_2 - a_1) \cdot t^2 = 2D.$$

$$t^2 = \frac{2D}{(a_2 - a_1)} > 0.$$

$$\begin{aligned} a_2 - a_1 &= \frac{T}{m_2} - \frac{T}{m_1} \\ &= \frac{(m_1 - m_2) \cdot T}{m_1 m_2} \end{aligned}$$

$$a_2 - a_1 = \frac{(m_1 - m_2) F}{2m_1 m_2}$$

$$t = \sqrt{\frac{4m_1 m_2 D}{(m_1 - m_2) F}}$$

↓

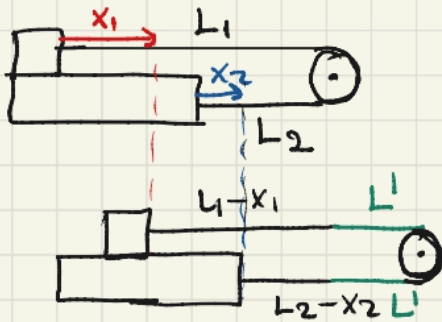
take the positive root.

Remark

Observe that if $m_1 = m_2$ $t \rightarrow \infty$, and if $m_1 < m_2$ $t = \sqrt{-(\dots)}$ $\rightarrow$ imaginary.

↳ So if $m_1 \geq m_2$, m_2 never catches m_1 .

d) Find the acceleration of the pulley with respect to the ground. (7 pts)



$L = L_1 + L_2 \rightarrow$ doesn't change.

$$L = L_1 - x_1 + L_2 - x_2 + 2L' = L_1 + L_2$$

$$L' = \frac{x_1 + x_2}{2}$$

$x_p \rightarrow$ displacement of pulley

$$a_p = \frac{d^2 x_p}{dt^2} = \frac{a_1 + a_2}{2} = \frac{(m_1 + m_2) F}{4 m_1 m_2}$$

$\downarrow$
acceleration of the pulley.

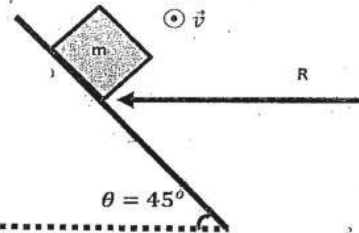
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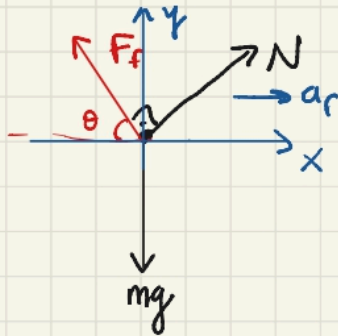
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Q4-(25 pts) A car of mass m is going around a circular turn of radius R over a wet concrete road, that is banked at an angle of $\theta=45^\circ$ with respect to the ground (cross section of the road is in the figure). The coefficient of static friction between the rubber tires and the road can be taken as $\mu=0.5$. You may neglect the kinetic friction.



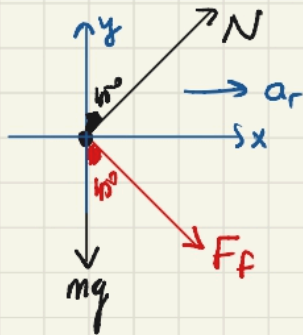
a) Sketch the free body diagram of the car if it just starts to slip down.



Remark:

Crucial part is to show friction upwards.

b) Sketch the free body diagram of the car if it just starts to slip up.



Remark:

Crucial part is to show friction downward.

c) If the car does not want to slip, what are the minimum and maximum allowed speeds?

$$a_y = 0.$$

For $v_{\max}$ we need to look at the Free Body Diagram in part b).

$$m a_y = F_y^{\text{tot}} = N \cdot \cos 45^\circ - mg - F_f \cdot \cos 45^\circ = 0.$$

where, $F_f = N \cdot \mu$.

$$\hookrightarrow N \cdot \sin 45^\circ \cdot \left(\frac{1-\mu}{1/2} \right) = mg \Rightarrow \boxed{N \cdot \sin 45^\circ = \frac{2mg}{1}} \quad (*)$$

Also we have:

$$a_x = \frac{F_x^{\text{tot}}}{m} = \frac{N \cdot \sin 45^\circ + F_f \cdot \sin 45^\circ}{m} = a_x = a_r = \frac{v_{\max}^2}{R}$$

Use (*)

$$\hookrightarrow \frac{v_{\max}^2}{R} = \frac{N \sin 45^\circ \cdot \left(\frac{1+\mu}{1/2} \right)}{m} = \frac{2mg \cdot 3/2}{m} = 3g$$

$$\hookrightarrow \boxed{v_{\max} = \sqrt{3gR}}$$

$\rightarrow$ For $v_{\min}$ we should use FBD in part a).

Again we want: $a_y = 0$, $a_x = a_r$

$$m a_y = F_y^{\text{tot}} = F_f \cdot \sin 45^\circ + N \cdot \sin 45^\circ - mg = 0.$$

$$F_f = \mu \cdot N \rightarrow \frac{3}{2} N \cdot \sin 45^\circ = mg.$$

$$\hookrightarrow \boxed{N \cdot \sin 45^\circ = \frac{2mg}{3}}.$$

$$a_x = \frac{F_x^{\text{tot}}}{m} = \frac{N \cdot \cos 45^\circ - F_f \cos 45^\circ}{m} = \frac{v_{\min}^2}{R} = a_r = a_x.$$

$$\hookrightarrow \frac{\frac{1}{2} \cdot N \cdot \cos 45^\circ}{m} = \frac{g}{3} = \frac{v_{\min}^2}{R}$$

$$\hookrightarrow v_{\min} = \frac{\sqrt{3gR}}{3}$$

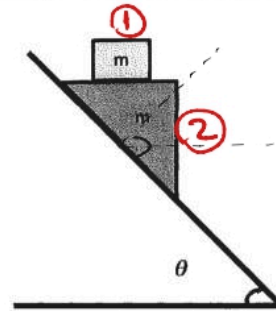
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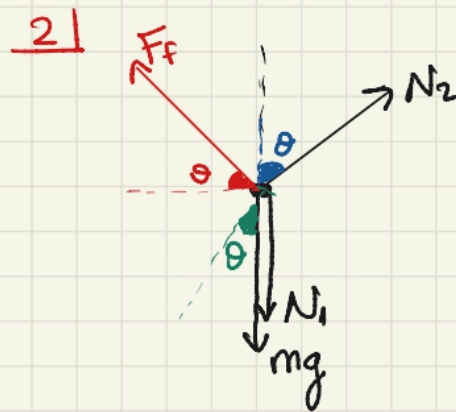
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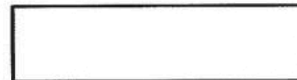
Q3-(25 pts) A triangular block with an exactly horizontal upper surface can slide down an inclined plane. There is a rectangular block on top of the triangular one as shown in the figure. There is no friction between the two blocks, but the coefficient of kinetic and static friction between the triangular block and the inclined plane is μ . Both blocks have mass m , and the inclination angle of the plane is θ .



a) Draw the free body diagrams of the blocks.



b) How is the magnitude of the acceleration of the rectangular block (a_r) related to the magnitude of the acceleration of the triangular block (a_t)?



Imagine triangular block moved a distance d .

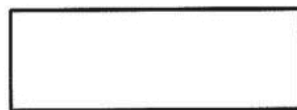
} rectangular block will move a distance $d \cdot \sin \theta$.

$$\hookrightarrow \frac{\Delta x_r}{\Delta x_t} = \sin \theta$$

Since we know $\Delta X_t = \frac{a_t \cdot t^2}{2}$
 $\Delta X_r = \frac{a_r \cdot t^2}{2}$

$$\hookrightarrow \frac{\Delta X_r}{\Delta X_t} = \frac{a_r}{a_t} = \sin \theta \rightarrow \boxed{a_r = a_t \cdot \sin \theta}$$

c) Find a_t and a_r .



$$a_r = \frac{mg - N_1}{m} = a_t \cdot \sin \theta \quad \hookrightarrow mg - N_1 = m \cdot a_t \cdot \sin \theta \quad \underline{(1)}$$

a_t is along the inclination,

$$m \cdot a_t = F_{\text{tot}} = mg \cdot \sin \theta + N_1 \cdot \sin \theta - F_f \quad \underline{(2)}$$

where $F_f = N_2 \cdot \mu$.

where $N_2 = mg \cdot \cos \theta + N_1 \cdot \cos \theta$.

Then, $F_f = mg \cdot \mu \cdot \cos \theta + N_1 \cdot \mu \cdot \cos \theta$.

Then, (2) becomes:

$$\begin{aligned} m a_t &= mg \cdot (\sin \theta - \mu \cos \theta) + N_1 \cdot (\sin \theta - \mu \cos \theta) \\ &= (mg + N_1) \cdot (\sin \theta - \mu \cos \theta) \end{aligned}$$

$$\hookrightarrow mg + N_1 = \frac{m a_t}{\sin \theta - \mu \cos \theta}$$

$$(1) \quad mg - N_1 = m a_t \cdot \sin \theta$$

$$mg + N_1 = \frac{m a_t}{\sin \theta - \mu \cdot \cos \theta}$$

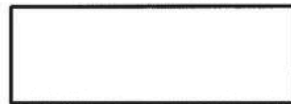
+

$$2mg = m a_t \left(\sin \theta + \frac{1}{\sin \theta - \mu \cdot \cos \theta} \right)$$

$$\hookrightarrow a_t = \frac{2g}{\sin \theta + \frac{1}{\sin \theta - \mu \cdot \cos \theta}}$$

$$\hookrightarrow a_r = a_t \cdot \sin \theta$$

d) For what values of μ blocks do not move at all?



$$\mu \geq \tan \theta.$$

notice that $m \cdot a_t = (mg + N_1) \cdot (\sin \theta - \mu \cdot \cos \theta)$

if $\mu > \tan \theta$ $\sin \theta - \mu \cdot \cos \theta \leq 0 \Rightarrow a \leq 0$.

But it means friction is accelerating our object in the direction of motion. Which is impossible.

for $\mu > \tan \theta \Rightarrow a \neq 0, a = 0$.



10. Fundamentals of Work and Energy

Work and Its Definition

WORK (Integral Form): Work is defined as the integral of the force along the path of displacement,

$$W_f = \int \mathbf{F} \cdot d\mathbf{l} \quad \text{or} \quad W = \int F \cos(\alpha) dl.$$

- For a **constant force**, work simplifies to the dot product:

Constant Force Work:

$$W = \mathbf{F} \cdot \mathbf{d}.$$

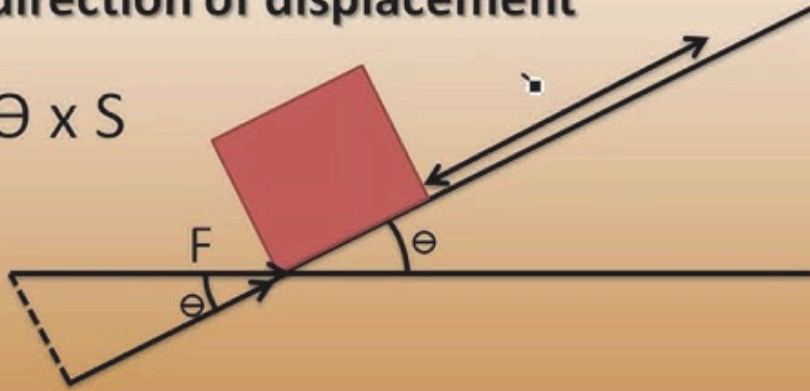
- For a **variable force**, the work is computed by integrating the force over the displacement:

Variable Force Work:

$$W = \int \mathbf{F}(x) dx.$$

Expression of Work – When force is in the direction of displacement

$$W = F \cos \theta \times S$$



Work done = Component of force in the direction of displacement $\times$ displacement

Latin's Academy

Work–Energy Theorem

WORK–ENERGY THEOREM: The net work done on an object is equal to its change in kinetic energy.

$$W_{\text{net}} = \Delta KE = KE_{\text{final}} - KE_{\text{initial}}.$$

- **Sign of Work:**

- **Positive Work:** When the force component is in the direction of displacement.
- **Negative Work:** When the force opposes the displacement.
- **Zero Work:** When the force is perpendicular to the displacement.

Springs (Hooke's Law) and Spring Work

HOOKE'S LAW: The restoring force of a spring is proportional to its displacement, given by

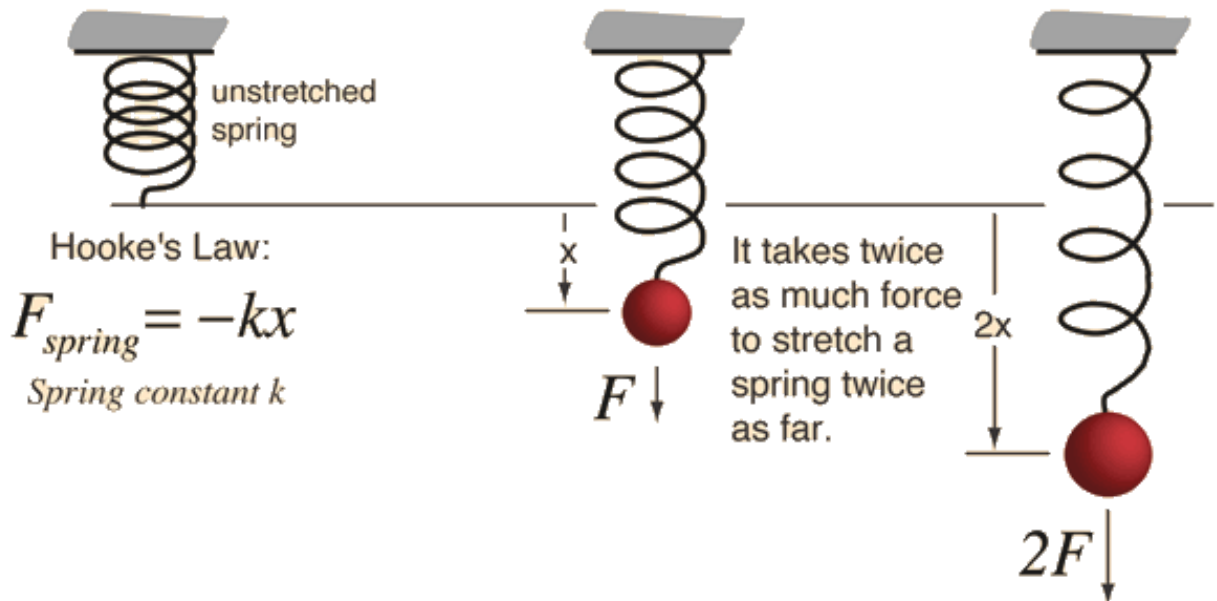
$$F_{\text{spring}} = -kx.$$

- **Work Done by a Spring (Variable Force):**

The work done by a spring as it is compressed or stretched from x_1 to x_2 is

$$W_{\text{spring}} = \int_{x_1}^{x_2} (-kx) dx = -\frac{1}{2}k(x_2^2 - x_1^2).$$

This work is related to the change in the spring's potential energy.



Gravitational Work and Inclined Planes

GRAVITATIONAL WORK on an Incline: When a mass m moves along an inclined plane with angle θ , the gravitational work depends on the component of weight along the plane:

$$W_g = mg \sin(\theta) \quad (\text{if motion is parallel to the incline}).$$

- **Other Forces Considered:**

- **Normal Force N :** Acts perpendicular to the plane.
- **Frictional Force f :** Given by $f = \mu N$, where μ is the coefficient of friction.

Additional Concepts

CONSTANT SPEED AND NET WORK: If an object moves at constant speed ($a = 0$), the net work done on the system is zero, implying no change in kinetic energy.

- **Units:**
 - Work and energy are measured in Joules (J).

GENERAL CONCEPT: Work represents how energy is added or removed from a system through the application of forces during motion.

Final Summary & Takeaways

- **Definition of Work:** Work is calculated as either the dot product of force and displacement (for constant forces) or as an integral for variable forces.
- **Work–Energy Theorem:** The net work done on an object is equal to its change in kinetic energy.
- **Spring Work:** Hooke’s law describes the restoring force of a spring, and the work done by a spring is proportional to the square of the displacement.
- **Gravitational Work on Inclines:** Calculated based on the component of weight along the incline, with friction and normal forces playing significant roles.
- **Practical Examples:** Numerical exercises illustrate how to compute work in different scenarios, reinforcing theoretical concepts with real-world applications.

Raw Notes

 Raw Notes



11. Work, Energy, and Power

Work and Its Definition

WORK (Integral Form): Work is defined as the integral of the force along the path of displacement,

$$W = \int \mathbf{F} \cdot d\mathbf{l} \quad \text{or} \quad W = \int F \cos(\theta) dl.$$

- For a **constant force**, the work simplifies to:

Constant Force Work:

$$W = \mathbf{F} \cdot \mathbf{d} = F d \cos(\theta)$$

- **Units:** Work is measured in Joules (J).

- **Graphical Interpretation:**

The area under a force vs. displacement curve represents the work done by a varying force.

Work–Energy Theorem

WORK–ENERGY THEOREM: The net work done on an object is equal to the change in its kinetic energy,

$$W_{\text{net}} = \Delta KE = KE_{\text{final}} - KE_{\text{initial}}.$$

- **Applicability:**

This theorem holds whether the force is constant or variable. For variable forces, the integral form of work is used.

- **Sign Considerations:**

- **Positive Work:** When the force component is in the direction of displacement.
- **Negative Work:** When the force opposes the displacement.
- **Zero Work:** When the force is perpendicular to the displacement.

Potential Energy (U)

Gravitational Potential Energy

GRAVITATIONAL POTENTIAL ENERGY: For an object of mass m at height y ,

$$U_g = mgy,$$

and a change in potential energy is given by $\Delta U = -W$ done by the conservative force.

Spring (Elastic) Potential Energy

SPRING POTENTIAL ENERGY: For a spring obeying Hooke's law,

$$F_{\text{spring}} = -kx,$$

the work done by the spring when compressed or stretched is

$$W_{\text{spring}} = \int (-kx) dx = -\frac{1}{2}k(x_2^2 - x_1^2),$$

which is equivalent to the change in spring potential energy.

Conservative vs. Non-Conservative Forces

CONSERVATIVE FORCES: Forces such as gravity and ideal spring forces that have an associated potential energy. *Characteristic:* The work done by these forces is path independent.

NON-CONSERVATIVE FORCES: Forces like friction and air resistance that do not have a recoverable potential energy component. *Effect:* They convert mechanical energy into other forms (e.g., heat), so mechanical energy is not conserved.

- **Energy Implications:**

- With only conservative forces, the total mechanical energy $E_{\text{mechanical}} = KE + U$ is conserved.
- The presence of non-conservative forces results in a net change in mechanical energy.

5. Conservation of Mechanical Energy

CONSERVATION OF MECHANICAL ENERGY: In the absence of non-conservative forces,

$$KE_i + U_i = KE_f + U_f.$$

- **Applications:**

- Determining maximum height in projectile motion.
- Analyzing motion in vertical loops or roller coaster problems.

Power

POWER: Power is defined as the rate at which work is done or energy is transferred, In cases where force and velocity are constant in direction,

$$P = \vec{F} \cdot \vec{v}.$$

- **Units:** Power is measured in Watts (W), where $1 \text{ W} = 1 \text{ J/s}$.

Final Summary & Takeaways

- **Work** is calculated as the dot product of force and displacement or, for variable forces, via integration.
- The **work-energy theorem** states that the net work done on an object equals its change in kinetic energy.
- **Potential energy** can be stored as gravitational or spring (elastic) potential energy, with the work done by conservative forces being recoverable.
- **Conservative forces** conserve mechanical energy, whereas **non-conservative forces** (like friction) convert mechanical energy into other forms.
- **Conservation of mechanical energy** is used to analyze systems where only conservative forces are acting.
- **Power** quantifies how quickly work is done or energy is transferred.

Raw Notes

 Raw Notes



12. Review: Work, Energy, and Power

Work & Energy Review Note

Key Concepts Recap

WORK: Defined as the integral of force along the displacement, $W = \int \mathbf{F} \cdot d\mathbf{l}$, or simplified for constant force to $W = F d \cos(\theta)$.

Units: Joules (J).

WORK-ENERGY THEOREM: The net work done on an object equals its change in kinetic energy, $W_{\text{net}} = \Delta KE$.

Energy Forms

- **Gravitational Potential Energy:**

$U_g = mgy$ with changes linked to work done by conservative forces.

- **Spring (Elastic) Potential Energy:**

Derived from Hooke's law with $F_{\text{spring}} = -kx$ leading to work $W_{\text{spring}} = -\frac{1}{2}k(x_2^2 - x_1^2)$.

Forces & Energy Conservation

CONSERVATIVE FORCES: Path-independent forces (e.g., gravity, ideal springs) that have recoverable potential energy.

NON-CONSERVATIVE FORCES: Forces (e.g., friction, air resistance) that convert mechanical energy to other forms, breaking energy conservation.

- **Conservation of Mechanical Energy:**

When only conservative forces are present, $KE + U$ remains constant.

Power

POWER: The rate at which work is done, expressed as $P = \vec{F} \cdot \vec{v}$.

Units: Watts (W), where $1 \text{ W} = 1 \text{ J/s}$.

Final Takeaways

- **Work** is computed via dot product or integration for variable forces.
- The **work-energy theorem** links net work to kinetic energy changes.
- **Potential energy** comes in gravitational and elastic forms.
- **Conservative forces** conserve mechanical energy, while non-conservative ones dissipate it.
- **Power** measures the rate of energy transfer.

Raw Notes

✿ Raw Notes



13. Energy Concepts Overview and Energy Diagram

Brief Overview of Key Energy Concepts

Kinetic Energy: The energy an object possesses due to its motion. It is given by

$$KE = \frac{1}{2}mv^2,$$

where m is mass and v is velocity.

Work-Energy Theorem: States that the net work performed on an object is equal to its change in kinetic energy:

$$W_{net} = \Delta KE.$$

Spring Work:

For a spring obeying Hooke's Law, the work done when compressing or stretching the spring is

$$W_{\text{spring}} = -\frac{1}{2}k(x_2^2 - x_1^2),$$

where k is the spring constant and x_1 and x_2 are the initial and final displacements respectively.

Conservative Forces vs. Non-conservative Forces:

- **Conservative Forces:**

These forces (e.g., gravity, ideal spring forces) have potential energy associated with them. The work done is independent of the path and is fully recoverable.

- **Non-conservative Forces:**

Examples include friction and air resistance. Their work depends on the path and typically dissipates energy as heat or sound.

Potential Energy (U) and Total Energy Work (W_t):

Potential energy is energy stored by an object due to its position or configuration. In conservative cases, the work done by or against conservative forces is stored as potential energy. In scenarios involving non-conservative forces, there is an additional loss or change in total mechanical energy.

Mechanical Energy: Defined as the sum of kinetic and potential energies:

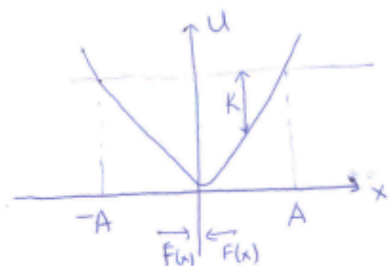
$$E_{\text{mechanical}} = KE + U.$$

In systems with only conservative forces, mechanical energy is conserved; when non-conservative forces act, mechanical energy changes accordingly.

Energy Diagrams

Energy diagrams : when a particle moves in 1D under the influence of a conservative force, a lot of insight can be gained by analyzing the graph of the potential energy function $U(x)$.

Consider the spring potential function $U(x) = \frac{1}{2} kx^2$

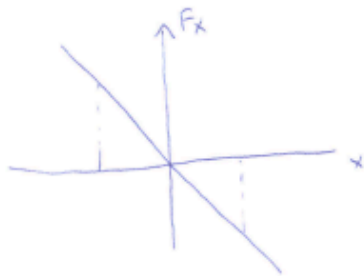


$$K + U = E$$

If the total energy of the particle is

$$E = \frac{1}{2} kA^2$$

$\Rightarrow$ the particle moves between $x = -A$ and $x = A$



$$F_x = -\frac{dU}{dx} = -kx$$

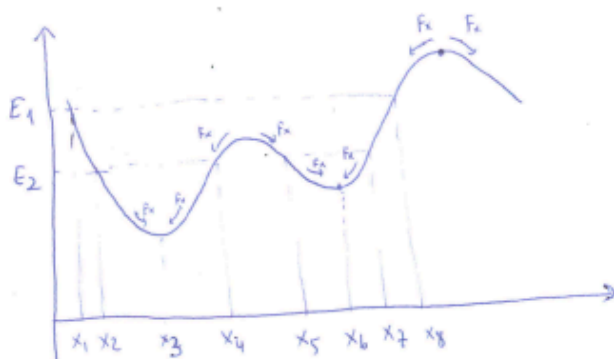
F_x is trying to bring the particle towards $x=0$.

$\Rightarrow x=0$ is a point of "stable equilibrium"

Any minimum in a potential energy curve is a stable equilibrium point.

* The particle will oscillate around the stable equilibrium.

Now consider a general $U(x)$:



when the particle is located on a local maxima $\frac{dU}{dx} = 0$, there is no force applied to the particle. A local max. represents an equilibrium point where the particle may stay forever.

However F_x is trying to move the particle away from the equilibrium point

$\Rightarrow$ A local maximum is an "unstable" equilibrium point

If total energy is $E_1 \Rightarrow$ particle oscillates between x_1 and x_8

$E_2 \Rightarrow$ particle oscillates either between x_2-x_4 or x_5-x_7

Raw Notes

 Raw Notes



14. Work & Kinetic Energy, Potential Energy & Energy Conservation

Work

- Constant force

$$W = \vec{F} \cdot \Delta \vec{r} = F d \cos \theta$$

- Variable force

$$W = \int_{\vec{r}_i}^{\vec{r}_f} \vec{F} \cdot d\vec{r}$$

Kinetic Energy & Work–Energy Theorem

- Kinetic energy

$$K = \frac{1}{2} m v^2$$

- Work–Energy Theorem

$$W_{\text{net}} = \Delta K = K_f - K_i$$

Conservative Forces & U

- A force $\vec{F}_c$ is conservative if $W_{A \rightarrow B} = -[U(B) - U(A)]$.
- **Gravity (near Earth)**

$$U_g(y) = m g y$$

- **Spring**

$$U_s(x) = \frac{1}{2} k x^2$$

Mechanical Energy Conservation

- Define

$$E_{\text{mech}} = K + U.$$

- **If only conservative forces act**

$$E_{\text{mech},i} = E_{\text{mech},f} \implies K_i + U_i = K_f + U_f.$$

- **If non-conservative work W_{nc} occurs**

$$W_{nc} = \Delta E_{\text{mech}}.$$



15. Linear Momentum & Impulse

Momentum & Impulse

- Linear momentum

$$\vec{p} = m \vec{v}$$

- Impulse

$$\vec{J} = \int_{t_i}^{t_f} \vec{F} dt = \Delta \vec{p}$$

Momentum Conservation

- In an **isolated** system (no net external impulse):

$$\sum \vec{p}_{\text{before}} = \sum \vec{p}_{\text{after}}.$$

Collisions

Collision type	Momentum	KE	Notes
Elastic	✓	✓	Objects “bounce”; no loss of kinetic energy.
Inelastic	✓	✗	Some KE → internal energy (heat, deformation).
Perfectly inelastic	✓	✗	Bodies stick together; maximal KE loss.

1D Two-Body Formulas

- **Perfectly inelastic**

$$v_f = \frac{m_1 v_{1i} + m_2 v_{2i}}{m_1 + m_2}$$

- **Elastic**

$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i} + \frac{2 m_2}{m_1 + m_2} v_{2i},$$

$$v_{2f} = \frac{2 m_1}{m_1 + m_2} v_{1i} + \frac{m_2 - m_1}{m_1 + m_2} v_{2i}.$$

Coefficient of Restitution

$$e = \frac{\text{relative speed after}}{\text{relative speed before}} = \frac{v_{2f} - v_{1f}}{v_{1i} - v_{2i}}.$$

- $e = 1$: perfectly elastic
 - $0 < e < 1$: partially inelastic
 - $e = 0$: perfectly inelastic
-



16. Rotation & Rigid Bodies

Rotational Kinematics

$$\omega = \frac{d\theta}{dt}, \quad \alpha = \frac{d\omega}{dt}$$

- Equations (constant α):

$$\begin{aligned}\omega &= \omega_0 + \alpha t, \\ \theta &= \theta_0 + \omega_0 t + \frac{1}{2} \alpha t^2, \\ \omega^2 &= \omega_0^2 + 2 \alpha (\theta - \theta_0).\end{aligned}$$

Moment of Inertia

$$I = \sum m_i r_i^2 \quad \text{or} \quad I = \int r^2 dm.$$

- **Parallel-axis theorem:**

$$I = I_{\text{CM}} + M d^2$$

Torque & Newton's 2nd Law (Rotation)

- **Torque:**

$$\vec{\tau} = \vec{r} \times \vec{F}.$$

- **Rotation analogue:**

$$\sum \tau_{\text{ext}} = I \alpha.$$

Rotational Work & Energy

- **Work:**

$$W = \int \tau d\theta.$$

- **Rotational KE:**

$$K_{\text{rot}} = \frac{1}{2} I \omega^2.$$

- **Rolling (no slipping):**

$$K = \frac{1}{2} m v_{\text{cm}}^2 + \frac{1}{2} I_{\text{cm}} \omega^2, \quad v_{\text{cm}} = \omega R.$$

Angular Momentum

- **Definition:**

$$\vec{L} = I \vec{\omega}.$$

- **Conservation:** if $\sum \tau_{\text{ext}} = 0$, then $\vec{L}$ is constant.

When Is What Conserved?

Quantity	Conserved if...
Linear momentum	$\sum \vec{F}_{\text{ext}} = 0$ (isolated system)
Mechanical energy	No net non-conservative work ($W_{nc} = 0$)
Kinetic energy	Only in elastic collisions or no NC forces
Angular momentum	$\sum \tau_{\text{ext}} = 0$



17. Rotational Kinematics and Dynamics

Objective & Scope

This note aims to cover the fundamental concepts of rotational motion in rigid bodies, specifically focusing on rotational kinematics and rotational dynamics.

Rotational Kinematics

Definitions

RIGID BODY: An object whose particles maintain constant distances from one another under the action of external forces.

ANGULAR DISPLACEMENT (θ): The angle through which a rigid body rotates about a fixed axis.

ANGULAR VELOCITY (ω): The rate of change of angular displacement, defined as:

$$\omega = \frac{d\theta}{dt}$$

ANGULAR ACCELERATION (α): The rate of change of angular velocity:

$$\alpha = \frac{d\omega}{dt}$$

Formulas for Rotational Kinematics

These equations describe rotational motion under constant angular acceleration:

- Angular velocity:

$$\omega = \omega_0 + \alpha t$$

- Angular displacement:

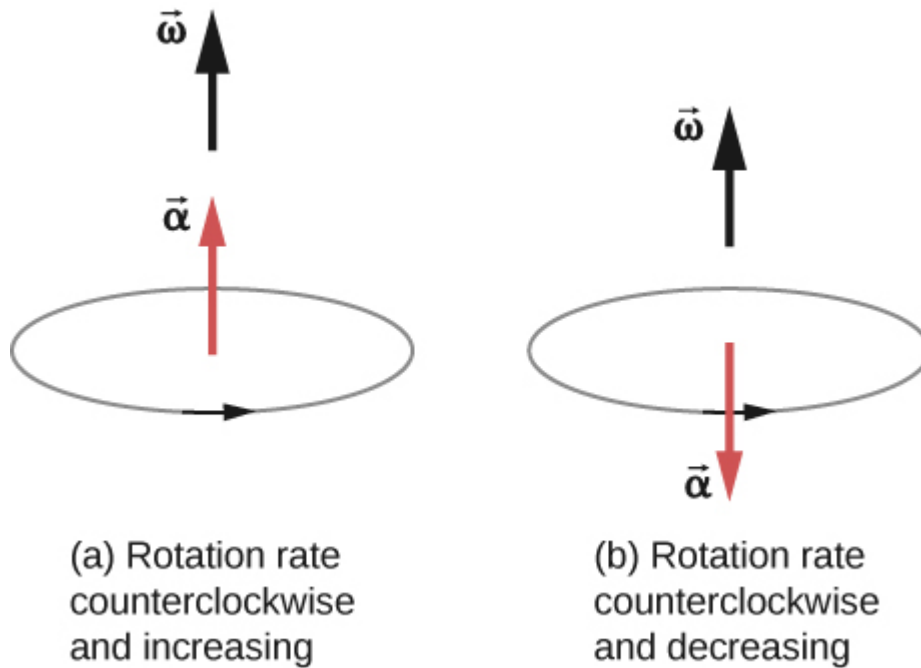
$$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$$

- Relationship between angular displacement and velocities:

$$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$$

- Relation between linear and angular quantities:

$$v = r\omega, \quad a_t = r\alpha, \quad a_c = \frac{v^2}{r} = r\omega^2$$



Rotational Dynamics

Definitions

TORQUE (τ): A measure of the rotational effect of a force, defined as:

$$\tau = rF \sin \phi$$

where:

- r is the distance from the axis of rotation,
- F is the magnitude of the force,
- ϕ is the angle between the force vector and the lever arm.

Newton's Second Law for Rotation

Analogous to linear motion ($F = ma$), rotational motion obeys:

$$\tau_{\text{net}} = I\alpha$$

- τ_{net} = net torque acting on the rigid body

- I = moment of inertia
- α = angular acceleration

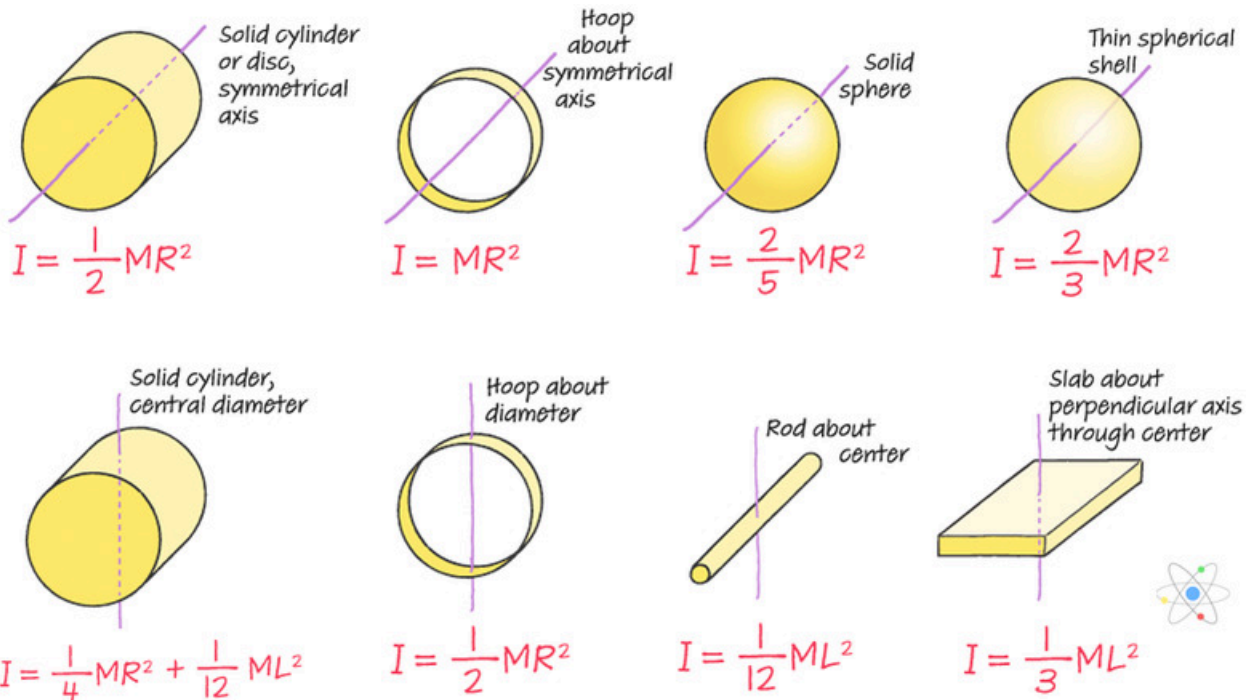
Moment of Inertia (I)

MOMENT OF INERTIA: A quantity expressing a rigid body's tendency to resist angular acceleration, calculated as:

$$I = \sum m_i r_i^2$$

Common Moments of Inertia:

- Solid sphere: $I = \frac{2}{5}MR^2$
- Solid cylinder (disk): $I = \frac{1}{2}MR^2$
- Hollow cylinder: $I = MR^2$
- Thin rod about center: $I = \frac{1}{12}ML^2$
- Thin rod about end: $I = \frac{1}{3}ML^2$



Example: Torque and Angular Acceleration

Example:

A force of 10 N is applied tangentially at the rim of a solid disk ($M = 4 \text{ kg}$, $R = 0.5 \text{ m}$). Determine the angular acceleration.

Solution:

- Calculate torque:

$$\tau = rF = (0.5 \text{ m})(10 \text{ N}) = 5 \text{ N}\cdot\text{m}$$

- Moment of inertia for solid disk:

$$I = \frac{1}{2}MR^2 = \frac{1}{2}(4 \text{ kg})(0.5 \text{ m})^2 = 0.5 \text{ kg}\cdot\text{m}^2$$

- Angular acceleration:

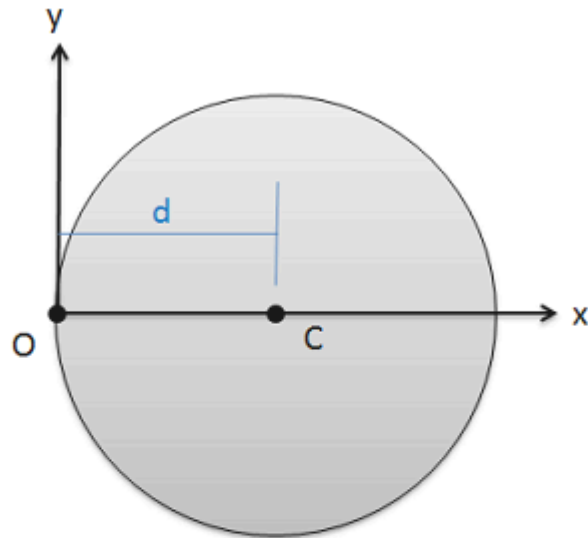
$$\alpha = \frac{\tau}{I} = \frac{5}{0.5} = 10 \text{ rad/s}^2$$

Parallel Axis Theorem

PARALLEL AXIS THEOREM: The moment of inertia I about any axis parallel to and a distance d away from an axis passing through the center of mass is given by:

$$I = I_{\text{CM}} + Md^2$$

- I_{CM} is the moment of inertia about the axis through the center of mass.
- M is the total mass of the rigid body.
- d is the perpendicular distance between the two parallel axes.



$$I_O = I_C + m d^2$$

Moment of
Inertia of
shape about
point O

Moment of
Inertia of
Shape about
Centroid (C)

Added Moment of
Inertia due to
distance between O
and C

Final Summary & Key Takeaways

- Understand clearly the difference between angular displacement, velocity, and acceleration.
 - Rotational kinematic equations parallel those of linear motion.
 - Torque is essential in rotational dynamics and is analogous to force in linear motion.
 - Moment of inertia is critical to analyzing rotational motion and depends on mass distribution.
-



18. Rotational Energy, Angular Momentum, and Equilibrium

Objective & Scope

This note expands upon rotation and rigid body concepts, covering rotational kinetic energy, angular momentum, conservation laws, and rotational equilibrium.

Rotational Energy

Rotational Kinetic Energy

ROTATIONAL KINETIC ENERGY: Energy due to rotation, given by:

$$K_{\text{rot}} = \frac{1}{2} I \omega^2$$

Work-Energy Principle in Rotational Motion

The work-energy theorem for rotation is given as:

$$W_{\text{net}} = \Delta K_{\text{rot}} = \frac{1}{2} I (\omega_f^2 - \omega_i^2)$$

Angular Momentum

Definition

ANGULAR MOMENTUM (L): A measure of rotational inertia in motion, defined as:

$$L = I\omega$$

For a particle or object in linear motion relative to a point:

$$L = r \times p = rmv \sin \phi$$

Conservation of Angular Momentum

CONSERVATION OF ANGULAR MOMENTUM: If no external torque acts on a system, the total angular momentum remains constant:

$$L_i = L_f \quad (\text{if } \tau_{\text{net}} = 0)$$

Example: Conservation of Angular Momentum

Example:

An ice skater spins with angular velocity 2 rad/s and moment of inertia $3 \text{ kg}\cdot\text{m}^2$. If she reduces her moment of inertia to $1.5 \text{ kg}\cdot\text{m}^2$, what is her new angular velocity?

Solution:

Using conservation of angular momentum:

$$I_i\omega_i = I_f\omega_f$$

- Substitute values:

$$(3)(2) = (1.5)\omega_f$$

- Solve for ω_f :

$$\omega_f = \frac{6}{1.5} = 4 \text{ rad/s}$$

Rotational Equilibrium and Stability

Conditions for Equilibrium

A rigid body is in rotational equilibrium if:

- Net external torque is zero:

$$\tau_{\text{net}} = 0$$

Types of Equilibrium

- **Stable equilibrium:** Displacement increases potential energy (returns to original position).
- **Unstable equilibrium:** Displacement decreases potential energy (moves away).
- **Neutral equilibrium:** Displacement neither increases nor decreases potential energy.

Rolling Without Slipping

ROLLING WITHOUT SLIPPING: A condition where a rolling object's linear velocity v is directly related to its angular velocity ω by:

$$v = r\omega$$

- v is the linear (translational) velocity of the center of mass.
- r is the radius of the rolling object.
- ω is the angular velocity.

Final Summary & Key Takeaways

- Rotational kinetic energy and angular momentum are crucial concepts analogous to their linear counterparts.
 - Angular momentum is conserved when no external torque acts on a system.
 - Understanding conditions for rotational equilibrium helps solve practical engineering and physical problems.
-



19. Rotational Kinematics & Dynamics

Rotational Kinematics

Angular Quantities

Angular Displacement θ : Angle through which a body rotates about a fixed axis.

Angular Velocity ω : Rate of change of angular displacement, $\omega = \frac{d\theta}{dt}$.

Angular Acceleration α : Rate of change of angular velocity, $\alpha = \frac{d\omega}{dt}$.

Kinematic Relations (constant α)

Relation:

$$\omega = \omega_0 + \alpha t$$

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$$

$$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$$

Linear–Angular Link:

$$v = r \omega,$$

$$a_{\text{tangential}} = r \alpha,$$

$$a_{\text{centripetal}} = r \omega^2.$$

Rotational Dynamics

Torque & Newton's Second Law

Torque τ : Tendency of a force $\mathbf{F}$ to rotate an object about an axis,

$$\tau = r F \sin \phi,$$

where r is lever arm and ϕ is angle between $\mathbf{F}$ and lever arm.

Rotational Law:

$$\sum \tau_{\text{ext}} = I \alpha,$$

analog of $F_{\text{net}} = m a$.

Moment of Inertia

Definition: $I = \int r^2 dm$, quantifying resistance to angular acceleration.

Parallel-Axis Theorem: If I_{CM} is about a centroidal axis, then about a parallel axis a distance d away,

$$I = I_{\text{CM}} + M d^2.$$

Common Forms:

- Solid disk about center: $I = \frac{1}{2} M R^2$
- Solid sphere about center: $I = \frac{2}{5} M R^2$
- Thin rod about center: $I = \frac{1}{12} M L^2$

Example:

A solid disk of mass 4 kg and radius 0.5 m has a tangential force $F = 10 \text{ N}$ applied at its rim.

Torque: $\tau = R F = 0.5 \times 10 = 5 \text{ N} \cdot \text{m}$

Moment of Inertia: $I = \frac{1}{2}(4)(0.5)^2 = 0.5 \text{ kg} \cdot \text{m}^2$

Angular Acceleration: $\alpha = \tau/I = 5/0.5 = 10 \text{ rad/s}^2$

Rotational Work & Energy

Work–Energy Principle

Rotational Work: $dW = \tau d\theta$.

Rotational Kinetic Energy:

$$K_{\text{rot}} = \frac{1}{2} I \omega^2.$$

Work–Energy Theorem:

$$W_{\text{net}} = \Delta K_{\text{rot}}.$$

Angular Momentum & Conservation

Angular Momentum L : For a rigid body about a fixed axis,

$$L = I \omega.$$

Conservation: If $\sum \tau_{\text{ext}} = 0$, then L is constant.

Example:

An ice skater spins at $\omega_i = 2 \text{ rad/s}$ with $I_i = 3 \text{ kg} \cdot \text{m}^2$. She pulls in her arms, reducing I_f to $1.5 \text{ kg} \cdot \text{m}^2$.

New angular speed:

$$I_i \omega_i = I_f \omega_f \implies \omega_f = \frac{I_i \omega_i}{I_f} = \frac{3 \times 2}{1.5} = 4 \text{ rad/s}.$$

Equilibrium & Rolling

Rotational Equilibrium: Net external torque zero, $\sum \tau_{\text{ext}} = 0$.

Rolling Without Slipping:

Relationship between translational velocity v and angular velocity ω :

| $v = r \omega.$

Final Summary & Takeaways

- Rotational kinematics mirror linear motion with θ, ω, α .
 - Torque and moment of inertia govern rotational acceleration: $\sum \tau = I \alpha$.
 - Energy methods use $K_{\text{rot}} = \frac{1}{2} I \omega^2$ and work–energy.
 - Angular momentum conservation applies when no external torque acts.
 - Rolling without slipping links v and ω .
 - **Common Mistakes:** Omitting $\sin \phi$ in torque, forgetting I dependence on axis, or neglecting r in rolling condition.
-



20. Rotation & Dynamics of Rigid Bodies

Rotation of Rigid Bodies

Rigid Body: An object whose internal distances remain constant during motion.

Axis of Rotation: The fixed line about which all points in the body move in circular paths.

Angular Displacement θ : The angle of rotation measured from a reference line.

Angular Velocity ω :

$$\omega = \frac{d\theta}{dt}.$$

Angular Acceleration α :

$$\alpha = \frac{d\omega}{dt}.$$

Linear–Angular Relationship: For a point at distance r from the axis:

$$v = r \omega, \quad a_{\text{tangential}} = r \alpha, \quad a_{\text{centripetal}} = r \omega^2.$$

Dynamics of Rotational Motion

Torque τ : The rotational analogue of force,

$$\tau = r F \sin \phi,$$

where ϕ is the angle between $\mathbf{F}$ and the lever arm. $\mathbf{F}$ and the lever arm.

Moment of Inertia I : The rotational analogue of mass:

$$I = \int r^2 dm,$$

quantifying a body's resistance to angular acceleration.

Rotational Form of Newton's Second Law:

$$\sum \tau_{\text{ext}} = I \alpha.$$

Rotational Kinetic Energy:

$$K_{\text{rot}} = \frac{1}{2} I \omega^2.$$

Angular Momentum:

$$L = I \omega,$$

conserved when $\sum \tau_{\text{ext}} = 0$.

Examples

Solid Disk under Constant Torque

A solid disk (mass M , radius R , $I = \frac{1}{2}MR^2$) experiences torque τ .

- Angular acceleration: $\alpha = \tau/I$.

- After time t : $\omega = (\tau/I) t$.

Ice Skater Pulling In Arms

Initial I_i, ω_i ; final I_f smaller when arms in.

Conservation: $I_i \omega_i = I_f \omega_f \implies \omega_f = \frac{I_i}{I_f} \omega_i$.

Final Summary & Takeaways

- Rigid bodies rotate about fixed axes with ω and α .
 - Torque and moment of inertia govern rotational acceleration: $\tau = I\alpha$.
 - Energy methods use $K_{\text{rot}} = \frac{1}{2}I\omega^2$.
 - Angular momentum $L = I\omega$ is conserved in absence of net torque.
 - **Common Mistakes:** Confusing torque direction, omitting the $\sin \phi$ factor, or miscomputing I for composite bodies.
-



21. Angular Momentum

Definition for a Particle

Angular Momentum ($\mathbf{L}$) of a particle of mass m with position $\mathbf{r}$ and momentum $\mathbf{p} = m\mathbf{v}$ about a chosen origin:

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}.$$

Angular Momentum of a Rigid Body

Total Angular Momentum: For a rigid body rotating about a fixed axis with angular velocity $\boldsymbol{\omega}$ and moment of inertia I :

$$\mathbf{L} = I\boldsymbol{\omega}.$$

More generally, $\mathbf{L} = \sum_i \mathbf{r}_i \times (m_i \mathbf{v}_i)$ or $\mathbf{L} = \mathbf{I}\boldsymbol{\omega}$ in tensor form.

Relation to Torque

Torque–Angular Momentum Theorem:

$$\sum \boldsymbol{\tau}_{\text{ext}} = \frac{d\mathbf{L}}{dt},$$

where $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$. A net external torque changes the system's angular momentum.

Conservation of Angular Momentum

Conservation Law: If $\sum \boldsymbol{\tau}_{\text{ext}} = 0$, then

$$\mathbf{L} = \text{constant}.$$

Applies to isolated systems, underpinning phenomena from planetary motion to figure skating.

Examples

Example (Particle in Central Force):

A particle moves under a central force $\mathbf{F} = f(r) \hat{r}$.

$\mathbf{r} \times \mathbf{F} = 0 \Rightarrow \mathbf{L} \text{ constant} \Rightarrow$ motion confined to a plane and equal areas in equal times.

Example (Ice Skater):

An ice skater with moment of inertia I_i and angular speed ω_i pulls in her arms to reduce to I_f .

$$I_i \omega_i = I_f \omega_f \implies \omega_f = \frac{I_i}{I_f} \omega_i.$$

Final Summary & Takeaways

- $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ for particles; $\mathbf{L} = I \boldsymbol{\omega}$ for rigid bodies.
- $\sum \boldsymbol{\tau} = d\mathbf{L}/dt$ links torque to change in angular momentum.
- **Conservation** holds when external torques vanish, explaining fixed-plane motion and speed-up in contracting systems.
- **Common Mistake:** Neglecting sign or direction in the cross product; forgetting that only external torques break conservation.



22. Periodic Motion

Definitions & Key Concepts

Periodic Motion: Motion that repeats itself in equal intervals of time.

Period (T): The time required for one complete cycle of motion.

Frequency (f): Number of cycles per unit time, $f = 1/T$.

Angular Frequency (ω): Rate of phase change, $\omega = 2\pi f = 2\pi/T$.

Simple Harmonic Motion (SHM)

Governing Equation

SHM: A restoring force proportional to displacement:

$$F = -k x.$$

From Newton's second law,

$$m\ddot{x} + kx = 0 \implies \ddot{x} + \omega^2 x = 0,$$

where $\omega = \sqrt{k/m}$.

General Solution

$$x(t) = A \cos(\omega t + \phi),$$

with amplitude A and phase constant ϕ .

Period & Frequency

$$T = \frac{2\pi}{\omega}, \quad f = \frac{\omega}{2\pi}.$$

Energy in SHM

Kinetic Energy:

$$K = \frac{1}{2} m \dot{x}^2.$$

Potential Energy:

$$U = \frac{1}{2} k x^2.$$

Total Energy: Constant,

$$E = K + U = \frac{1}{2} k A^2.$$

Examples

Example: Mass–Spring System

A block of mass m on a spring of constant k :

$$T = 2\pi \sqrt{\frac{m}{k}}, \quad \omega = \sqrt{\frac{k}{m}}.$$

If $m = 0.5 \text{ kg}$, $k = 200 \text{ N/m}$, then

$$\omega = 20 \text{ rad/s}, \quad T = \frac{2\pi}{20} \approx 0.314 \text{ s}.$$

Example: Simple Pendulum (Small Angle)

Length L , gravity g :

$$\ddot{\theta} + \frac{g}{L}\theta = 0 \quad \implies \quad T = 2\pi\sqrt{\frac{L}{g}}.$$

For $L = 1 \text{ m}$, $g = 9.8 \text{ m/s}^2$,

Final Summary & Takeaways

- **Periodic motion** repeats in time with well-defined T , f , and ω .
 - **SHM** arises when a linear restoring force acts; governed by $\ddot{x} + \omega^2 x = 0$.
 - **Energy** oscillates between kinetic and potential forms, with constant total E .
 - **Mass-spring** and **pendulum** illustrate how system parameters set the oscillation period.
 - **Common Mistake:** Applying SHM formulas beyond their small-angle or linear-force approximations.
-



23. Damped and Driven Harmonic Motion

1. Damped Harmonic Motion

1.1 Equation of Motion

The motion of a mass–spring system with linear damping is governed by

$$m \ddot{x} + c \dot{x} + k x = 0$$

- m : mass (kg)
- c : damping coefficient (kg/s)
- k : spring constant (N/m)

Divide by m :

$$\ddot{x} + 2\zeta\omega_0 \dot{x} + \omega_0^2 x = 0, \quad \text{where} \quad \omega_0 = \sqrt{\frac{k}{m}}, \quad \zeta = \frac{c}{2\sqrt{mk}}.$$

1.2 Damping Regimes

Define the damping ratio ζ :

- **Underdamped** ($\zeta < 1$)
- **Critically damped** ($\zeta = 1$)
- **Overdamped** ($\zeta > 1$)

1.2.1 Underdamped ($\zeta < 1$)

Solution:

$$x(t) = A e^{-\zeta\omega_0 t} \cos(\omega_d t + \phi), \quad \omega_d = \omega_0 \sqrt{1 - \zeta^2}.$$

- **Damped period:** $T_d = 2\pi/\omega_d$
- **Envelope:** $x_{\max}(t) = A e^{-\zeta\omega_0 t}$

1.2.2 Critically Damped ($\zeta = 1$)

Solution:

$$x(t) = (A + B t) e^{-\omega_0 t}$$

1.2.3 Overdamped ($\zeta > 1$)

Solution:

$$x(t) = C e^{-\omega_0(\zeta + \sqrt{\zeta^2 - 1}) t} + D e^{-\omega_0(\zeta - \sqrt{\zeta^2 - 1}) t}.$$

1.3 Energy Dissipation

- **Total energy** decays exponentially:

$$E(t) = \frac{1}{2} k A^2 e^{-2\zeta\omega_0 t}.$$

- **Time constant:** $\tau = 1/(\zeta\omega_0)$.

After $t = \tau$, energy has dropped to $e^{-2} \approx 14\%$ of initial.

2.1 Equation of Motion

Add a sinusoidal driving force $F(t) = F_0 \cos(\omega t)$:

$$m \ddot{x} + c \dot{x} + k x = F_0 \cos(\omega t).$$

2.2 Steady-State Solution

Assume particular solution:

$$x_p(t) = A(\omega) \cos(\omega t - \phi(\omega)).$$

- **Amplitude:**

$$A(\omega) = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\zeta\omega_0\omega)^2}}.$$

- **Phase lag:**

$$\tan \phi(\omega) = \frac{2\zeta\omega_0\omega}{\omega_0^2 - \omega^2}.$$

2.3 Resonance & Quality Factor

- **Resonant frequency** (underdamped):

$$\omega_r = \omega_0 \sqrt{1 - 2\zeta^2} \quad (\zeta < 1/\sqrt{2}).$$

- **Max amplitude** at resonance:

$$A_{\max} \approx \frac{F_0}{2m\omega_0^2\zeta}.$$

- **Quality factor:**

$$Q = \frac{1}{2\zeta} = \frac{\omega_0}{\Delta\omega},$$

where $\Delta\omega$ is the half-power bandwidth.

3. Phase Relationships & Power

- **Low ω ($\omega \ll \omega_0$):** x nearly in phase with F .

- **At resonance:** phase lag $\phi = 90^\circ$.
- **High ω ($\omega \gg \omega_0$):** x lags by $\phi \rightarrow 180^\circ$.
- **Average power absorbed:**

$$\langle P \rangle = \frac{1}{2} c \omega^2 A^2(\omega).$$

4. Key Takeaways

- **Damping** reduces amplitude and energy ($E \propto e^{-2\zeta\omega_0 t}$).
 - **Critical damping** yields fastest return without oscillation.
 - **Resonance** maximizes steady-state amplitude; high Q systems are sharply tuned.
 - **Phase lag** transitions from 0° to 180° as driving frequency increases.
-



24. Gravitation

1. Introduction to Gravitation

Gravitation is a fundamental interaction that causes every particle of matter in the universe to attract every other particle. It governs the motion of planets, stars, galaxies, and governs phenomena on Earth such as falling objects, tides, and the behavior of satellites.

- Historically, Isaac Newton formulated a quantitative law of universal gravitation in the 17th century.
- Later, Johannes Kepler, based on Tycho Brahe's observational data, formulated his three empirical laws of planetary motion.
- In the early 20th century, Einstein's General Theory of Relativity provided a deeper understanding of gravitation as curvature of spacetime; however, for PHYS101, we focus on Newtonian gravity and Kepler's laws.

2. Newton's Law of Universal Gravitation

Statement: Every point mass attracts every other point mass in the universe with a force that is:

1. Directly proportional to the product of their masses.
2. Inversely proportional to the square of the distance between their centers.

Mathematically,

$$F = G \frac{m_1 m_2}{r^2}$$

where:

- F is the magnitude of the gravitational force between two masses (in newtons, N).
- m_1, m_2 are the masses of the two objects (in kilograms, kg).
- r is the distance between the centers of the two masses (in meters, m).
- G is the universal gravitational constant.

2.1 Gravitational Constant (G)

- The gravitational constant G has an experimentally determined value:

$$G = 6.67430 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$$

- All gravitational interactions in classical mechanics use this same constant.

2.2 Weight vs. Mass

- **Mass (m)**
 - Scalar quantity.
 - Measure of the amount of matter in an object (in kg).
 - Invariant (does not change with position).
- **Weight (W)**
 - Vector quantity.
 - The gravitational force exerted on an object by a massive body (e.g., Earth).
 - At Earth's surface, $W = m g$ where $g \approx 9.81 \text{ m/s}^2$.
 - Weight varies with local gravitational acceleration; mass remains constant.

3. Gravitational Field and Potential

3.1 Gravitational Field Strength (g)

- Definition: The gravitational field strength at a point in space is the gravitational force per unit mass experienced by a small test mass placed at that point.

$$\mathbf{g}(\mathbf{r}) = \frac{\mathbf{F}_g}{m_{\text{test}}} \quad \left[\text{units: N/kg or m/s}^2 \right]$$

- For a point mass M located at the origin, the gravitational field at a distance r is radial and given by:

$$g(r) = \frac{F}{m_{\text{test}}} = G \frac{M}{r^2}$$

directed toward the mass M .

- Near the Earth's surface ($r \approx R_{\oplus}$), $g \approx 9.81 \text{ m/s}^2$.

3.2 Gravitational Potential (V)

- Gravitational potential $V(\mathbf{r})$ at a point is defined as the gravitational potential energy per unit mass at that point (with reference at infinity taken to be zero).

$$V(r) = -G \frac{M}{r}$$

- The negative sign indicates that work must be done (by an external agent) to bring a test mass from infinity to a distance r .
- Relationship between field and potential:

$$\mathbf{g}(\mathbf{r}) = -\nabla V(\mathbf{r})$$

In spherical symmetry,

$$g(r) = -\frac{dV}{dr} = G \frac{M}{r^2}$$

3.3 Gravitational Potential Energy (U)

- For two point masses m and M separated by distance r , the gravitational potential energy is:

$$U(r) = -G \frac{M m}{r}$$

- If two masses are infinitely far apart ($r \rightarrow \infty$), $U(\infty) = 0$. As they come together, $U(r)$ becomes more negative.
- In the context of a single mass m in the gravitational field of a much larger mass M :

$$U(r) = m V(r) = -G \frac{M m}{r}$$

4. Motion Under Gravity Near Earth's Surface

4.1 Uniform Gravitational Field Approximation

For motions close to Earth's surface where $r \approx R_{\oplus}$, gravitational acceleration g can be treated as uniform:

$$g \approx 9.81 \text{ m/s}^2 \quad (\text{constant})$$

4.2 Equations of Motion

For an object thrown or dropped in a uniform gravitational field (vertical motion):

- Displacement:

$$y(t) = y_0 + v_{0y} t - \frac{1}{2} g t^2$$

- Velocity (vertical component):

$$v_y(t) = v_{0y} - g t$$

- Kinematic equation (no explicit time):

$$v_y^2 = v_{0y}^2 - 2 g (y - y_0)$$

(Here, upward is taken as positive; if downward is chosen positive, sign of g changes accordingly.)

4.3 Escape Velocity

- Definition: The minimum initial speed needed for a mass m to "escape" from the gravitational pull of a mass M (i.e., reach infinity with zero kinetic energy).
- Starting at radius $r = R$ with speed v_{esc} , energy conservation gives:

$$\frac{1}{2} m v_{\text{esc}}^2 - G \frac{M m}{R} = 0 \implies v_{\text{esc}} = \sqrt{\frac{2 G M}{R}}$$

- For Earth ($M = M_{\oplus}$, $R = R_{\oplus}$):

$$v_{\text{esc, Earth}} = \sqrt{2 g R_{\oplus}} \approx 11.2 \times 10^3 \text{ m/s}$$

4.4 Circular Orbit Velocity

- For a small mass m in a circular orbit of radius r around a larger mass M :

1. Centripetal force requirement:

$$\frac{m v^2}{r} = G \frac{M m}{r^2}$$

2. Solve for orbital speed v :

$$v_{\text{orbit}} = \sqrt{\frac{G M}{r}}$$

- The corresponding orbital period T is:

$$T = \frac{2\pi r}{v_{\text{orbit}}} = 2\pi \sqrt{\frac{r^3}{GM}}$$

5. Kepler's Laws of Planetary Motion

Johannes Kepler (1609–1619) empirically determined three laws describing planetary motion about the Sun. These can be derived from Newton's law of gravitation and laws of motion.

5.1 First Law (Law of Ellipses)

- **Statement:** The orbit of every planet is an ellipse, with the Sun at one of the two foci.
- **Elliptical Geometry:**
 - Semi-major axis: a
 - Semi-minor axis: b
 - Distance between center and each focus: c , where $c^2 = a^2 - b^2$.
 - For an ellipse, any point P on the ellipse satisfies:

$$\overline{PF_1} + \overline{PF_2} = 2a$$

- Eccentricity:

$$e = \frac{c}{a}, \quad 0 \leq e < 1$$

5.2 Second Law (Law of Equal Areas)

- **Statement:** A line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time.
- **Mathematical Formulation:**
 - Areal velocity $\frac{dA}{dt}$ is constant:

$$\frac{dA}{dt} = \frac{L}{2m}$$

where L is the angular momentum of the planet (constant).

- Equivalently, for a small time Δt , the area swept ΔA is constant for all equal Δt .

5.3 Third Law (Harmonic Law)

- **Statement:** The square of the orbital period T of a planet is proportional to the cube of the semi-major axis a of its orbit:

$$T^2 \propto a^3$$

More precisely, for a small body orbiting the Sun (mass $M_{\odot}$):

$$T^2 = \frac{4\pi^2}{GM_{\odot}} a^3$$

where:

- T : orbital period (in seconds).
- a : semi-major axis of orbit (in meters).
- $M_{\odot}$: mass of the Sun.

5.4 Derivation of the Third Law from Newtonian Gravitation

Assumptions:

- A small mass m ($m \ll M$) orbits a much larger mass M (e.g., a planet around the Sun) in a nearly elliptical—or for simplicity, circular—orbit of radius $r \approx a$.
- Centripetal force** provided by gravity:

$$\frac{m v^2}{r} = G \frac{M m}{r^2} \implies v = \sqrt{\frac{G M}{r}}.$$

- Orbital period** T for a circular orbit:

$$T = \frac{2\pi r}{v} = 2\pi r \sqrt{\frac{r}{G M}} = 2\pi \sqrt{\frac{r^3}{G M}}.$$

- Square both sides:**

$$T^2 = \left(2\pi \sqrt{\frac{r^3}{G M}}\right)^2 = \frac{4\pi^2}{G M} r^3.$$

For elliptical orbits, replace r with the semi-major axis a :

$$T^2 = \frac{4\pi^2}{G M} a^3.$$

6. Orbital Mechanics: Circular and Elliptical Orbits

6.1 Circular Orbits

- Orbital Speed** (repeated from Section 4.4):

$$v_{\text{circ}} = \sqrt{\frac{G M}{r}}$$

- Orbital Period** (also repeated):

$$T = 2\pi \sqrt{\frac{r^3}{G M}}$$

- Total Mechanical Energy** of a circular orbiting mass m :

$$E_{\text{total}} = K + U = \frac{1}{2} m v_{\text{circ}}^2 - \frac{G M m}{r}.$$

Substitute $v_{\text{circ}}^2 = \frac{G M}{r}$:

$$E_{\text{total}} = \frac{1}{2} m \frac{G M}{r} - \frac{G M m}{r} = -\frac{1}{2} \frac{G M m}{r}.$$

- Negative total energy indicates a bound orbit.

6.2 Elliptical Orbits

- For an ellipse with semi-major axis a and semi-minor axis b :
 - Distance from focus (Sun) to a point at true anomaly θ :

$$r(\theta) = \frac{a(1 - e^2)}{1 + e \cos \theta}.$$

where e is the eccentricity.

- **Orbital Speed** at any point r in an elliptical orbit (vis-viva equation):

$$v(r) = \sqrt{G M \left(\frac{2}{r} - \frac{1}{a} \right)}.$$

- At perihelion ($r = a(1 - e)$), speed is maximum.
- At aphelion ($r = a(1 + e)$), speed is minimum.

- **Specific Orbital Energy** (energy per unit mass) for an elliptical orbit:

$$\varepsilon = \frac{v^2}{2} - \frac{G M}{r} = -\frac{G M}{2 a}.$$

Thus, for any r on the ellipse, ε remains constant.

7. Applications and Examples

7.1 Satellite Period Calculation

Problem: Calculate the orbital period of a satellite in a circular orbit at 300 km above Earth's surface.

1. **Given Data:**

- Earth's radius $R_{\oplus} = 6.371 \times 10^6$ m.
- Satellite altitude above surface: $h = 300 \times 10^3$ m.
- Earth's mass $M_{\oplus} = 5.972 \times 10^{24}$ kg.
- Gravitational constant $G = 6.67430 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$.

2. **Orbit Radius:**

$$r = R_{\oplus} + h = 6.371 \times 10^6 + 3.00 \times 10^5 = 6.671 \times 10^6 \text{ m}.$$

3. **Orbital Period T :**

$$T = 2\pi \sqrt{\frac{r^3}{G M_{\oplus}}}.$$

- Compute r^3 :

$$r^3 = (6.671 \times 10^6)^3 \approx 2.970 \times 10^{20} \text{ m}^3.$$

- Then:

$$T = 2\pi \sqrt{\frac{2.970 \times 10^{20}}{6.67430 \times 10^{-11} \times 5.972 \times 10^{24}}} = 2\pi \sqrt{\frac{2.970 \times 10^{20}}{3.985 \times 10^{14}}} = 2\pi \sqrt{7.455 \times 10^5} \approx 2\pi \times 864.0 \approx 5,428 \text{ s}$$

- Convert to minutes:

$$T \approx \frac{5,428}{60} \approx 90.5 \text{ minutes}.$$

7.2 Escape Velocity from Earth

Problem: Determine the escape velocity from Earth's surface.

1. **Given Data:**

- Earth mass: $M_{\oplus} = 5.972 \times 10^{24} \text{ kg}$.
- Earth radius: $R_{\oplus} = 6.371 \times 10^6 \text{ m}$.
- $G = 6.67430 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$.

2. **Formula:**

$$v_{\text{esc}} = \sqrt{\frac{2 G M_{\oplus}}{R_{\oplus}}}.$$

3. **Calculation:**

- Numerator:

$$2 G M_{\oplus} = 2 \times 6.67430 \times 10^{-11} \times 5.972 \times 10^{24} \approx 7.973 \times 10^{14}.$$

- Divide by $R_{\oplus}$:

$$\frac{7.973 \times 10^{14}}{6.371 \times 10^6} \approx 1.252 \times 10^8.$$

- Take square root:

$$v_{\text{esc}} \approx \sqrt{1.252 \times 10^8} \approx 1.119 \times 10^4 \text{ m/s} = 11.19 \text{ km/s}.$$

8. Summary of Key Equations

1. **Newton's Law of Universal Gravitation:**

$$F = G \frac{m_1 m_2}{r^2}$$

2. **Gravitational Field Strength:**

$$g(r) = G \frac{M}{r^2}$$

3. **Gravitational Potential:**

$$V(r) = -G \frac{M}{r}$$

4. **Gravitational Potential Energy:**

$$U(r) = -G \frac{M m}{r}$$

5. **Escape Velocity:**

$$v_{\text{esc}} = \sqrt{\frac{2 G M}{R}}$$

6. **Circular Orbit Velocity:**

$$v_{\text{circ}} = \sqrt{\frac{G M}{r}}$$

7. **Orbital Period (Circular Orbit):**

$$T = 2\pi \sqrt{\frac{r^3}{G M}}$$

8. **Kepler's Third Law (Harmonic Law):**

$$T^2 = \frac{4\pi^2}{G M} a^3$$

9. **Vis-Viva Equation (General Orbital Speed):**

$$v(r) = \sqrt{G M \left(\frac{2}{r} - \frac{1}{a} \right)}.$$
